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REPORTS, PAPERS, DISCUSSIONS, AND MEMOIRS

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VIRTUAL WORK: A RESTATEMENT

By Hardy Cross,* M. Am. Soc. C. E.

Synopsis

The fundamental relations that determine the distortions produced in structures by internal strains may be stated in the mathematical language of the calculus and of geometry and in the engineering terminology of moment, shear, product of inertia, and moment of inertia. These theorems are, however, most readily correlated by use of the principle of virtual work. The paper restates this principle, emphasizing that the reactions to the imaginary external resistance are themselves purely imaginary and in no way necessarily related to the supports of the structure. It then illustrates in part the very broad application of the principle.

Most of the theorems developed are not new. Indeed, the writer is not sure that any of them are entirely new, nor is he here especially concerned with their lack of novelty. The paper may serve a useful purpose in bringing into logical sequence a group of geometrical principles which are essential to the study of statically indeterminate structures. If it has any such value, the omission of references to the work of many who have helped to develop or illuminate the principle of virtual work, may be pardoned.

Virtual Work—Usual Statement

Although essentially a principle of geometry, the principle of virtual work has proved the most powerful tool ever applied to the analysis of statically indeterminate structures. An examination of American textbooks leads to the belief that the broadest application of the principle has not been properly emphasized.

As usually presented, the principle may be stated somewhat as follows:† Assume any rigid body acted on by a force and reactions and let it be required to determine the deflection, Δ, in the direction, $a\,b$, of any point, a. Consider, first, the stretch—call it δ—of any differential fiber. Now considering a unit hypothetical resistance to motion at a to produce a stress, u, in the fiber, then the external work will be $1 \times \Delta$ and the internal work will be $u \times \delta$. Then,

$$\Delta = u\,\delta \ldots\ldots\ldots\

Each fiber produces its independent deflection and, therefore,

$$\text{Total } \Delta = \Sigma u \delta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

If the internal hypothetical stress resists the distortion, the internal work will be positive and the external movement opposite in direction to the hypothetical load. The same result follows from the more convenient rule, which treats lengthening and tension as positive, and indicates a positive displacement in the direction of the hypothetical resistance when the algebraic summation, $\Sigma u \delta$, is positive.

Reference has been made to the stretch of a differential fiber—the argument would have applied equally well to the shearing distortion of a differential cube, to the rotational distortion of a differential cylinder, to internal angular displacement, or to any other internal distortion.

Also, as Professor Swain has pointed out,* it is possible to deal with external rotation at a instead of translation. The hypothetical unit resistance is then a unit moment. Then,

$$\text{Rotation at } a = \Sigma u \delta,$$

in which, u is the stress in each small particle due to a unit moment at a and δ is the strain actually existing in the particle.

VIRTUAL WORK—RESTATEMENT

As the equations of virtual work just stated involve no work done by the reactions from the hypothetical unit resistance it must have been assumed that they do no work and hence that they do not move. This is commonly—although not always—stated. A more correct statement, however, is that these reactions do not have any motion which appears as a part of the described displacement, and hence the reactions to the hypothetical unit resistance are such as to fix in direction the line with reference to which the desired displacement is measured and to fix in position any assumed point on this line.

Commonly this line of reference is the line joining the supports of the structure, but it might equally well be either (1) the line joining any two points on the deflected structure or (2) any tangent to it. In the former case the unit load or moment acts on a structure simply supported at the two given points; in the latter, the structure is fixed, or cantilevered, at the point of tangency of the fixed tangent.

The general principle deduced, therefore, may be stated, as follows: To find with reference to any line of the distorted structure and with reference to any point on that line the displacement produced at any other point by given internal distortions, consider a unit resistance to such displacement applied externally at the given point, the structure being held stable by reactions fixing the point and the line of reference. Find the internal resistances produced by this imaginary external resistance. The sum of the products of the internal distortions multiplied by the imaginary internal resistances is the desired displacement.

The internal distortions may be due to any condition of stress and the strains may be either elastic or plastic, or they may be due to temperature or

* *Transactions*, Am. Soc. C. E., Vol. LXXXIII (1919–20), p. 622 *et seq.*

inaccurate workmanship. The external movement may be either of rotation or of translation. Rotation may be measured with reference to any line in the deflected structure. Translation may be measured in any direction with reference to any given line in the deflected structure and any given point on that line.

Virtual Work Applied to Trusses

Thus presented the broad statement of the relations of external movements to internal distortions leads directly to the usual theorems dealing with slopes and deflections. In trusses it takes the familiar form, $\sum \frac{s\,u\,l}{A\,E}$ (the properties of the member being s, the stress; l, the length; and A, the area; E being the modulus of elasticity), except that u should be defined with reference to the fixed or reaction points. Thus, it is possible to find the deflection of any panel point, A, with reference to a line through any other two panel points, B and C, by considering the hypothetical unit load applied at A, the truss being supported at B and C. Either B or C may be taken as fixed, the other point being, for hypothetical loads, on rollers. Similarly, the relative deflection of opposite corners of any quadrilateral of a truss may be found by taking $\Sigma\,\delta\,u$, in which, u is the stress in any bar due to a unit load at one corner in the direction of the other when the truss is supported at the other corner. This is a familiar problem in internal indetermination.

Virtual Work Applied to Beams

In the case of beams the displacements due to moment only are:

$$\Delta = \int M'\,d\,\phi \qquad (3)$$

in which, Δ is the external relative movement of any point, $d\,\phi$ is the rotation (produced by the bending moments) in any differential length, and M' is the imaginary moment over this differential length produced by a unit resistance to the external movement.

If the deflection at any point from a chord of a structure, due to moments only, is to be found, the values, M', will be identical with the ordinates to an influence line for bending moment at this point on the chord. From this it follows that the deflections of any slightly curved line away from its chord equal the bending moments on the chord, treated as a beam, simply supported at its ends, due to the angle changes as loads. This is applicable in any case to a beam, a floor line, or a truss chord, if the angle changes can be computed. From it the moment-area theorems of Mohr and Greene may be shown to follow directly, or they may be taken as direct corollaries of the principle of virtual work, as will be shown.

The angle change in a differential length of the axis of a beam, if plane sections remain plane, is the strain of an outer fiber, divided by the distance of this fiber from the neutral axis. The strain of the outer fiber is $\frac{f}{E}\,d\,l$, in

which, f is the intensity of stress in the outer fiber. Then, $d\,\phi = \frac{f}{E\,y}\,d\,l$, in which, y is the distance from outer fiber to neutral axis.

In all the theorems dealing with moment-areas, the term, $\frac{f}{y}$, may be substituted for $\frac{M}{I}$. Sometimes this is convenient. It also makes possible a clearer understanding of the deflection of reinforced concrete beams and, in general, gives a clearer view of the assumptions that are involved in the analysis of indeterminate structures.

If the beam formula applies, then,

$$f = \frac{M\,y}{I};\ d\,\phi = \frac{f}{E\,y}\,d\,l = \frac{M}{E\,I}\,d\,l$$

Hence,

$$\Delta = \int M'\,M\,\frac{d\,l}{E\,I} \qquad (4)$$

If applied to a part of a beam containing a frictionless hinge, this expression is evidently indeterminate, since at the hinge both M and I are zero. Moment-area theorems are, therefore, indeterminate for those parts of a beam in which a hinge occurs unless the change of angle at the hinge is computed independently.

The Reciprocal Theorem

If the expressions, $\sum \frac{s\,u\,l}{A\,E}$ for trusses and $\int M'\,M\,\frac{d\,l}{E\,I}$ for beams, be used to find absolute displacements due to loads on the structure, the interchangeability of the terms, s and M, which are due to the loads, with u and M', respectively, which are due to the hypothetical external unit resistances to displacement, indicates at once the general theorem of reciprocal displacements. If "displacement" is interpreted in a general sense as either linear or angular, and "load" in a general sense as either force or moment, then in any structure the displacement at A due to a load at B has the same value as the displacement at B due to a load at A, provided that both at A and at B the force and the displacement are of the same nature, linear displacement corresponding to force along its line of action, and rotation corresponding to moment.

This theorem is useful principally in interpreting as influence ordinates those displacements of the load line of a structure which would be produced by an imaginary unit internal distortion corresponding to the internal stress for which the influence line is desired.

Greene's Theorems Derived Directly by Virtual Work

If the relative rotation at one point on a beam referred to the tangent at another point is desired, consider the beam cantilevered from one point and loaded with a unit moment at the other point. Then, $M' = \text{constant} = 1$, and,

$$\int \frac{M\,M'}{E\,I}\,d\,l = \frac{1}{E\,I} \times (\text{area under moment curve between the points of reference}) \qquad (5)$$

If, however, the deflection of the second point with reference to a tangent from the first is wanted, consider the beam to be loaded and cantilevered from the first point, with a unit load at the second. Then M' equals the distance of each section from the second point and $\int \left(\frac{M}{E\,I}\,d\,l\right) M'$ is evidently $\frac{1}{E\,I}$ times the statical moment of the area of the moment curve about the second point.

Mohr's Theorems

In the case of a beam on fixed supports the change of slope at any point, A, relative to the line joining these supports may be found by applying a unit moment at the point, the beam being simply supported at the fixed points. The moment curve for M' will be found to be identical with an influence line for shear at A. Hence,

$$\int \frac{M\,M'}{E\,I}\,d\,l = \int \frac{M\,d\,l}{E\,I} \times (\text{shear at } A \text{ due to a unit load at each section})$$

$$= \int \left(\text{shears at } A \text{ due to each } \frac{M\,d\,l}{E\,I}\right)$$

$$= \text{shear at } A \text{ on a simple beam due to the } \frac{M}{E\,I}\text{-curve considered as a load.} \quad (6)$$

For the deflection at A, on a simply supported beam, $D\,E$, apply a unit load at A. The curve for M' is then identical with an influence line for bending moment at A. Hence, the deflection of A is the bending moment at A due to the $\frac{M}{E\,I}$-curve considered as a load.

The method herein indicated of establishing the identity of the curve for M' with the influence line for some easily evaluated function, such as shear or moment, offers a general method for stating theorems of slope and deflection in the language of structural engineering. By its use may be readily determined those analogies which form the foundation for what has come to be known as the method of conjugate beams.

Evidently, any line on the beam other than $D\,E$ may be considered as fixed. Thus, consider a beam bent as shown in Fig. 1. The deflection of Point A with reference to the line, $B\,C$—Δ_a in the diagram—is the bending moment at A produced by the $\frac{M}{E\,I}$-curve between B and C on a simply supported beam, $B\,C$. This is sometimes a convenient theorem.

Also, the slope of the tangent at A with reference to the line, $B\,C$, is the shear at A on the simply supported beam, $B\,C$, due to the $\frac{M}{E\,I}$-curve between B and C acting as a load.

Angle Changes in Trusses

Another interesting application of the principle of virtual work is in finding the change, $\Delta\alpha$, of any angle, α, of a triangle (Fig. 2) due to stresses in the three sides. Apply at A and C the elements of a unit couple, causing reactions such as to hold $A\,B$ fixed in direction. Tension and increase of angle are taken as positive. Then,

$$\Delta\alpha = \Sigma\,\delta\,u = \Sigma\,\frac{f\,l}{E}\,u$$

or,

$$E\,\Delta\alpha = \Sigma\,f\,l\,u \qquad (7)$$

in which, for any bar, f is the fiber stress, l, the length, and E the elastic modulus.

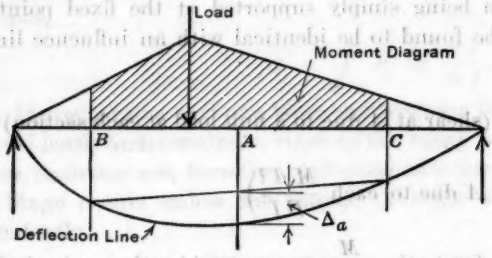

Fig. 1.

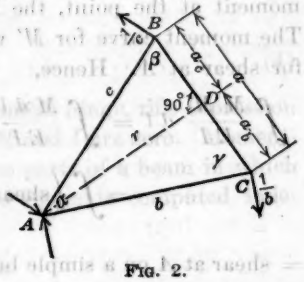

Fig. 2.

Resolving the forces at Joints B and C,

$$u_a = \frac{1}{r}\,;\quad u_b = -\frac{1}{b}\,\frac{a_b}{r}\,;\quad u_c = -\frac{1}{c}\,\frac{a_c}{r}$$

$$E\,\Delta\alpha = f_a\,a\,\frac{1}{r} - f_b\,b\,\frac{1}{b}\,\frac{a_b}{r} - f_c\,c\,\frac{1}{c}\,\frac{a_c}{r} = \frac{f_a\,a - (f_b\,a_b + f_c\,a_c)}{r}$$

$$= \left[f_a - \left(f_b\,\frac{a_b}{a} + f_c\,\frac{a_c}{a}\right)\right]\frac{a}{r} \qquad (8)$$

in which, f_a, f_b, f_c are fiber stresses and the other terms are as shown in Fig. 2. Equation (8) is probably the most convenient formula for the angle changes. It may be modified to:

$$E\,\Delta\alpha = f_a\,\frac{a_b}{r} + f_a\,\frac{a_c}{r} - f_b\,\frac{a_b}{r} - f_c\,\frac{a_c}{r}$$

$$= (f_a - f_b)\,\frac{a_b}{r} + (f_a - f_c)\,\frac{a_c}{r}$$

$$= (f_a - f_b)\cot\gamma + (f_a - f_c)\cot\beta \qquad (9)$$

Equation (9) is familiar to students of secondary stresses.

Evidently, the method here presented may be readily extended to the quadrilaterals which occur in subdivided trusses and K-trusses.

Slope Deflection

The application of this method to the special case of a beam or part of a beam loaded only with moments at its two ends gives directly either the end

slopes in terms of the end moments or the latter in terms of the former. In the first case given the loads on the beam, the reactions (end shears) are desired; in the second, the magnitude of the loads is to be determined for given reactions. By taking moments about B (Fig. 3), there follows,

$$\theta_a = \frac{2}{3} M_a \frac{l}{2EI} - \frac{1}{3} M_b \frac{l}{2EI} = \frac{l}{6EI}(2M_a - M_b) \quad \ldots\ldots (10)$$

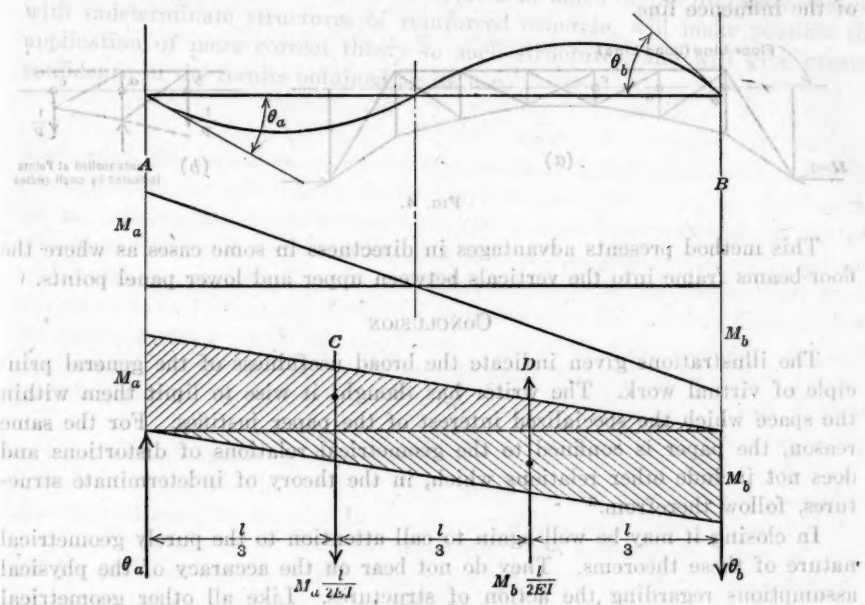

FIG. 3.

By taking moments about D,

$$\frac{M_a l}{2EI} = 2\theta_a + \theta_b$$

or,

$$M_a = \frac{2EI}{l}(2\theta_a + \theta_b) \quad \ldots\ldots (11)$$

Both forms of equations have long been used in evaluating secondary stresses. Equation (11), combined with the device of using as unknowns the rotations of parts of the structure and evaluating them by simultaneous equations of statics, has had an elaborate development in the literature of the "slope-deflection method."

MOMENT WEIGHTS

The principle of virtual work furnishes directly a method of computing moment weights, or, as some writers call them, "elastic weights", for determining the deflected load line, or influence line, for trusses. Assume that it is desired to draw the deflected load line for a unit load as shown (Fig. 4), that is, an influence line for horizontal reaction. The angle change at a may be computed by applying a unit moment resisting this angle change. This is

effected by applying loads as shown in Fig. 4 (*b*), acting at the points on the load line or floor as indicated by circles, and then computing $\Sigma \delta u$, in which, δ is the change in length of any bar due to the horizontal reaction, and u is the stress for the loading shown in Fig. 4 (*b*). These angle changes may then be treated as loads at the panel points and, when corrected for the deflection of the ends of the load line, the moment curve thus produced will have the shape of the influence line.

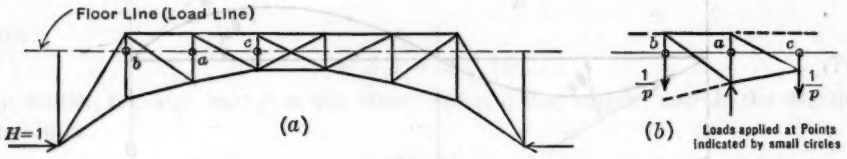

FIG. 4.

This method presents advantages in directness in some cases as where the floor-beams frame into the verticals between upper and lower panel points.

CONCLUSION

The illustrations given indicate the broad usefulness of the general principle of virtual work. The writer has thought it wise to limit them within the space which the specialized interest of the paper justifies. For the same reason, the paper is confined to the geometrical relations of distortions and does not include other relations which, in the theory of indeterminate structures, follow therefrom.*

In closing it may be well again to call attention to the purely geometrical nature of these theorems. They do not bear on the accuracy of the physical assumptions regarding the action of structures. Like all other geometrical relations, they are not open to dispute, nor are they proper subjects for experimental verification, and they are also subject to statement in an endless variety of forms or aspects of the relations involved. Familiarity with these relations is essential to facility in analyzing indeterminate structures, but neither ordinary nor extraordinary expertness in this field necessarily connotes either unusual or even common ability in designing structures or in understanding their action.

To apply these theorems to engineering structures, it is simply necessary either to show that for given values of the moments, shears, and thrusts, the strains can be predicted, or, if exact prediction is not possible, to determine by direct computation what error results from the inexactness. In analyzing indeterminate structures it is also to be noted that usually the relative and not the absolute values of these strains are in question.

Because of the definiteness of the moments, shears, and thrusts, strain measurements on statically determinate structures seem for this purpose more valuable and dependable than similar data derived from measurements on indeterminate frames. When it is established that, for any type of construction, these strains—or their relative values—can be definitely predicted, the

* For an interesting correlation of virtual work with the theorems of Castigliano, see "The Principle of Virtual Velocities and Its Application to the Theory of Elastic Structures", by Ernest Lamb, *Selected Engineering Papers No. 10*, Inst. C. E., December, 1923.

whole theory of indeterminate stress analysis follows from the relations of the distortions as a matter of geometry. In order to be of real value in designing indeterminate structures, however, the load-strain relations must be those for conditions approaching failure and not merely those that exist at working stresses. An understanding of these facts will make clearer the limitations of the theory of elasticity as applied in much of the literature dealing with indeterminate structures of reinforced concrete, will make possible the application of more correct theory to such structures, and will give greater confidence in the results obtained by its use.

STRESSES IN THICK ARCHES OF DAMS

By B. F. Jakobsen,* M. Am. Soc. C. E.

Synopsis

The stresses in arches considered as parts of an arch dam are determined on the assumptions that Poisson's ratio is zero and that the vertical downward stresses produced by the weight of the concrete and of the water acting on the up-stream face are negligible; it is further assumed that the modulus of elasticity is constant and that normal planes remain plane, so that deformations due to shear do not influence the distribution of the normal stresses due to bending. Making these the usual assumptions the writer finds formulas for stresses, which formulas are identical with those developed by William Cain, M. Am. Soc. C. E. This serves to check the method of development used in this paper.

Poisson's ratio is then assumed as finite and the structure is considered as an arch rather than as a straight beam, that is, the actual position of the neutral axis is found instead of, as usual, assuming it to coincide with the gravity axis, and the stress distribution is determined.

The introduction of Poisson's ratio and of the correct location of the neutral axis does not seem to influence the final results materially, except for short and thick arches. The influence of the vertical stress is found to be considerable for high dams.

The influence of temperature variations, swelling, and shrinkage is investigated; the formulas presented make it possible to assume different conditions for the up-stream and for the down-stream faces. This assumption is probably nearer the truth than that of a uniform change of volume for the entire section of an arch, as the up-stream face, when loaded, is water-soaked, whereas the down-stream face is not, or perhaps to a lesser degree, depending on the thickness of the arch. Also, the temperature of the up-stream face is likely to differ from that of the down-stream face. The influence of the part of the water pressure that is carried by the dam considered as a cantilever and the effect of pressure grouting are also investigated.

The stress determination made by the use of Lamé's formulas for thick hollow cylinders and of Airy's stress function is on the assumptions that the modulus of elasticity and Poisson's ratio remain constant, and does not involve any of the other usual assumptions previously mentioned. Judging from the thick and rather short arch investigated, the usual assumptions are

Note.—Written discussion on this paper will be closed with the May, 1926, *Proceedings*. When finally closed, the paper, with discussion in full, will be published in *Transactions*.

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fairly accurate for ordinary arches in arch dams, but the more nearly correct formulas here developed show a somewhat more advantageous stress distribution than previous formulas indicated.

It is shown that radial sections cannot remain plane and that the correct stresses cannot be found unless the deformations of the abutments are taken into account.

The influence of the variation of the modulus of elasticity on the stress distribution is discussed, and it is shown that this variation tends to make the stress distribution more uniform than if the modulus of elasticity were to remain constant. Tables and diagrams are given to facilitate computations.

The theory of the secondary arch is discussed and convenient approximate formulas are derived. Approximate formulas are also given for an arch with a variable thickness.

It is shown that the swelling of the up-stream face due to water-soaking produces an up-stream displacement of the arches and a down-stream deflection of the cantilever; the deflection of a gravity dam due to the deformations of the foundation is also investigated. These factors are shown to be of relatively great magnitude. The usual methods of dividing the load between the cantilevers and the arches, which neglect those factors, cannot lead to trustworthy results.

I.—Introduction

The Pacoima Dam, a constant-angle arch dam approximately 400 ft. high, is now under construction by the Los Angeles County Flood Control District.* Its estimated cost is $1 750 000 and the saving, compared to a gravity dam, is in excess of $1 000 000. Besides, the constant angle arch dam has a much larger factor of safety than the gravity form.

In investigating the Pacoima Dam with the Cain formulas in the usual manner, that is, neglecting the influence of the vertical stresses due to the weight of the concrete, tension is found in the thicker arches. In order to transmit this tension about $60 000 worth of reinforcing steel in place would be necessary. The question, therefore, arose as to whether this tension actually existed or whether it was to be attributed to inaccuracies in the theory used in determining the stresses. It will be appreciated that the proper determination of this question is of considerable economic importance, especially as the constant angle type offers opportunities for rational design, that the constant radius type cannot offer, and as it is likely to be increasingly used.

In discussing Professor Cain's methods the writer has shown† that these reduce to the formulas for straight beams when the radius approaches infinity and the central angle approaches zero. In deriving the ordinary bending formulas it is assumed (see Fig. 1), that $\sigma_y = 0$, or that Poisson's ratio,

* A license arrangement has been entered into with the patentee, The Constant Angle Arch Company, San Francisco, Calif., to furnish the design and occasional inspection during construction.

† "The Circular Arch Under Normal Loads," *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 254, and Professor Cain's closing discussion, p. 272.

$\frac{1}{m} = 0$. In reality, σ_y produces an elongation in the direction of σ_x which will modify the stress distribution. When the thickness of the beam is small compared to its length, as is generally the case, the influence of σ_y is quite negligible; but when the thickness is large compared to the length, the influence of σ_y becomes greater, and for a very short beam σ_y may be greater than σ_x. This is one reason why the Cain formulas cannot be expected to hold for very short arches and still be fairly accurate for the ordinary arches of arch dams. Another difficulty encountered in attempting to determine the stress distribution in very short arches is due to the restraining action of the abutments, which may influence this distribution throughout the entire arch. This difficulty is well known from the uncertainty in the determination of the stresses at the foundation of a gravity dam, where, unfortunately, they are greatest.

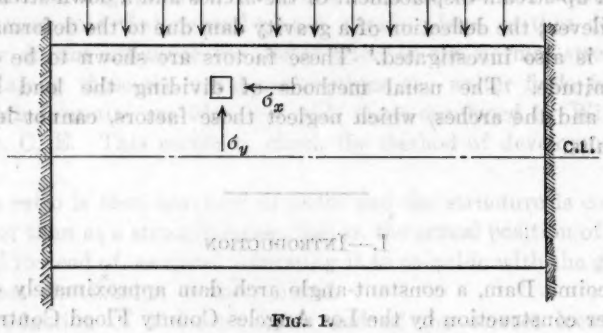

FIG. 1.

In an arch dam arch action and cantilever action exist simultaneously. To insure this double action, contraction joints should be provided at suitable intervals and these joints should be pressure-grouted after the concrete has set and cooled, that is, after most of the shrinkage has taken place and the temperature of the whole structure is near its minimum.* This provision also tends to eliminate internal stresses, as the contraction joints permit adjustments of the various parts of the dam. Furthermore, the pressure grouting increases the vertical downward stresses in the up-stream face, which is an advantage.

That some skill is necessary in order to insure a continuous structure is illustrated by the Mulholland Dam, completed in December, 1924, by the City Water Department of Los Angeles, Calif. The dam is about 200 ft. high and is a so-called gravity arch, in which all or a large part of the load is carried by the cantilever, provided no uplift exists. In July, 1925, while the air temperature was high, the dam had developed a number of cracks, two of which were large and could be traced on the down-stream face down to the foundation and on the up-stream face down to the water surface, which was about 30 ft. below the crest. There was not much leakage, probably because the concrete

* "Improving Arch Action in Arch Dams," by Lars R. Jorgensen, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. LXXXIII (1919–20), p. 314.

inside the dam had not had time to cool. It is hardly safe to assume that this structure can act as an arch dam, which requires a continuous structure.

The contraction of concrete due to setting may amount to as much as 0.0002 and the thermal effect should be added to this. If a modulus of elasticity of 2 000 000 be assumed, the shrinkage of 0.0002 is equivalent to a tensile stress of 400 lb. per sq. in., or if Poisson's ratio is equal to 0.125, a vertical compression of 3 200 lb. per sq. in. would be necessary to compensate for a lateral shrinkage of 0.0002. In the thicker part of a dam no doubt the shrinkage is much less than 0.002, and the water, being under considerable pressure in contact with the up-stream face, may also help to prevent the concrete from drying out. When the up-stream face is continually wet, a swelling of about 0.0001 may be expected.* Little is known regarding the volumetric changes of mass concrete, the individual designer usually making his own assumptions. The temperature variations inside a great mass of concrete† are known to be small after the chemical heat has been dissipated.

In designing an arch dam, it is convenient to determine the stresses due to the water pressure in two parts on the assumption that the arches and cantilevers divide the load and that no volumetric changes (due to temperature, swelling, and shrinkage) occur. Then some assumptions must be made regarding temperature variations, swelling, and shrinkage, depending on how the dam is built and whether or not pressure grouting is to be used.

It is planned to install in the Pacoima Dam some resistor type of cartridges with temperature coils, as developed by the U. S. Bureau of Standards. This will furnish information regarding the deformations due to stresses and the volumetric changes due to temperature variations, shrinkage in setting, effect of water-soaking, etc., will determine how effective the pressure grouting is and will suggest the best method of procedure for accomplishing this grouting. Proper pressure grouting is essential to insure co-operation between the cantilevers and the arches and prevent tension from developing in the arches.

The purpose of this paper is to study the stresses in thick arches including the influence of Poisson's ratio and of the vertical compression due to the weight of the concrete and the water acting vertically downward on the up-stream face.

II.—Derivation of Professor Cain's Formulas

Professor Cain makes two assumptions which, for short thick arches, are questionable, namely,

(1) That the rotation of the section of the arch brought about by the uniformly distributed normal stresses, that is, compression or tension, may be neglected; and

(2) That Poisson's ratio $m = 0$. The customary method of computing stresses in an arch, by the formula:

* "Shrinkage and Time Effects in Reinforced Concrete," by F. R. McMillan, M. Am. Soc. C. E., Univ. of Minnesota, *Studies in Engineering No. 3*; see, also, the writer's discussion in *Transactions*, Am. Soc. C. E., Vol. LXXXIV (1921), p. 103, and especially Fig. 36 on p. 104, reproduced from Professor McMillan's paper.

† "Temperature Changes in Mass Concrete," by Charles H. Paul, M. Am. Soc. C. E., and the late A. B. Mayhew, Assoc. M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. LXXIX (1915), p. 1225.

$$\sigma = \frac{P}{t} \pm 6\,\frac{M}{t^2} \quad \ldots\ldots\ldots\

up-stream face by p_a is negative. Displacements in the direction of X are also taken as positive and likewise displacements of the crown section toward the center, C, or in a down-stream direction, are taken as positive.

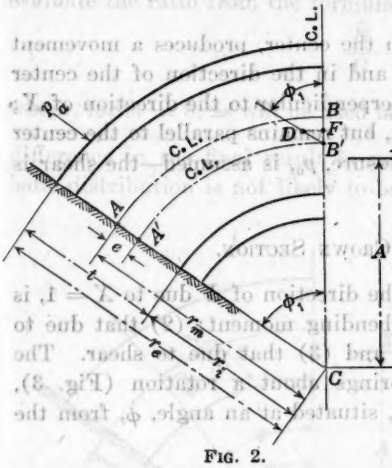

FIG. 2.

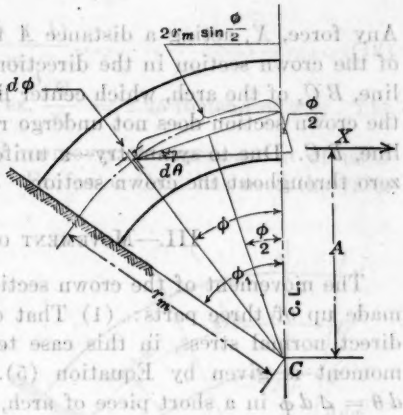

FIG. 3.

The displacement, $e = A\,A'$, in Fig. 2, computed from the ordinary cylinder formula, is,

$$e = \frac{p_a r_e}{E t} r_m = \frac{p_m r_m^2}{E t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which, p_m is the water pressure referred to the center line, so that

$$p_m = p_a \frac{r_e}{r_m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

p_a and E both being in pounds per square foot, and t, r_e, r_m, and e being in feet. The force, X, must pull the crown section a distance, $e \sin \phi_1 = D\,F$, in the direction of X. At the same time, X produces a movement at right angles to its own line of action; this will be considered later, when the deflection is to be determined.

In order to check Professor Cain's formulas, it is necessary to adhere to the assumptions made by him. That is, the ordinary cylinder formula must be used in determining the stresses and deformations in the cylinder due to water pressure, as that formula follows from the assumption that Poisson's ratio, $\frac{1}{m} = 0$, and the rotation of the crown section due to the uniformly distributed normal stresses must be neglected, that is, Professor Cain's Equation (6),*

$$\int_0^{\phi_1} M\,d\,\phi = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

must be used to determine A in Fig. 2.

The bending moment at any section of an arch due to $X = 1$, is,

$$M = A - r_m \cos \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

* *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 237.

M is taken as positive when it brings about an elongation in the extrados. From Equations (4) and (5),

$$A = r_m \frac{\sin \phi_1}{\phi_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Any force, X, acting a distance A ft. from the center, produces a movement of the crown section in the direction of X and in the direction of the center line, $B\,C$, of the arch, which center line is perpendicular to the direction of X; the crown section does not undergo rotation, but remains parallel to the center line, $B\,C$. Due to symmetry—a uniform pressure, p_a, is assumed—the shear is zero throughout the crown section.

III.—Movement of the Crown Section.

The movement of the crown section in the direction of X due to $X = 1$, is made up of three parts: (1) That due to bending moment; (2) that due to direct normal stress, in this case tension; and (3) that due to shear. The moment is given by Equation (5). It brings about a rotation (Fig. 3), $d\,\theta = \Delta\, d\,\phi$ in a short piece of arch, $r_m\, d\,\phi$, situated at an angle, ϕ, from the center line, and hence,

$$d\,\theta = \Delta\, d\,\phi = \frac{M}{E\,I}\, r_m\, d\,\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

in which, I is the moment of inertia. The displacement of the crown section in the direction of X due to the bending moment produced by $X = 1$ lb. acting on an arch slice 1 ft. thick, is then,

$$y_M = \int_0^{\phi_1} 2\, r_m \sin^2\left(\frac{\phi}{2}\right) d\,\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Introducing the value of M and $d\,\theta$ from Equations (5) and (7),

$$y_M = \int_0^{\phi_1} 2\, r_m \sin^2\left(\frac{\phi}{2}\right) \frac{A - r_m \cos \phi}{E\,I}\, r_m\, d\,\phi$$

and integrating, since $\sin^2\left(\frac{\phi}{2}\right) = \frac{1}{2}\,(1 - \cos \phi)$,

$$y_M = \frac{r_m^2}{E\,I}\left[A\,(\phi_1 - \sin \phi_1) - r_m\left(\sin \phi_1 - \frac{\phi_1}{2} - \frac{1}{4}\sin 2\,\phi_1\right)\right] \ldots (9)$$

The unit direct normal stress (in this case, tension) is (Fig. 4) $\frac{\cos \phi}{t}$, and the displacement of the crown section in the direction of X due to the tension produced by $X = 1$, is,

$$y_N = \int_0^{\phi_1} \frac{\cos^2 \phi}{E\,t}\, r_m\, d\,\phi = \frac{r_m}{2\,E\,t}\left[\phi_1 + \frac{1}{2}\sin 2\,\phi_1\right] \ldots\ldots (10)$$

The unit shear stress is $\frac{\sin \phi}{t}$ and the displacement of the crown section in the direction of X due to the shear stress produced by $X = 1$, is,

$$y_S = 2.88 \int_0^{\phi_1} \frac{r_m}{E\,t} \sin^2 \phi\, d\,\phi = 2.88\, \frac{r_m}{2\,E\,t}\left[\phi_1 - \frac{1}{2}\sin 2\,\phi_1\right] \ldots (11)$$

This is on the assumption that the shear is distributed across the section according to a parabola and that $\frac{E}{G} = 2.4$. It would be more correct to evaluate the ratio from the formula,

$$\frac{E}{G} = \frac{2\,(m+1)}{m} \qquad (12)$$

which, for $m = 8$, as will be used later, gives, $\frac{E}{G} = 2.25$ instead of 2.4. The difference in the final result is small and moreover the assumption of parabolic distribution is not likely to be exact. Professor Cain has used the value

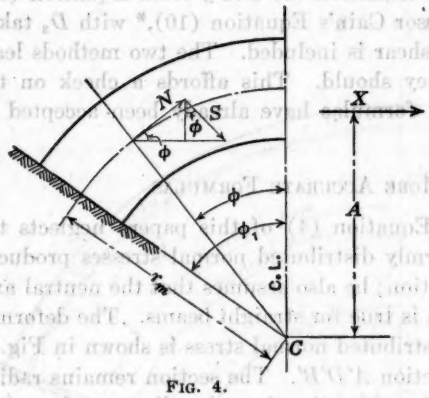

FIG. 4.

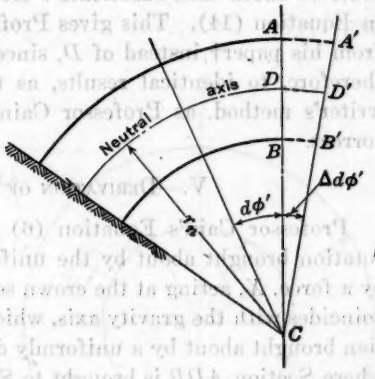

FIG. 5.

2.4, which will be used herein. The total displacement of the crown in the direction of X due to $X = 1$, is, therefore:

$$y = y_M + y_N + y_S$$

or,

$$y = \frac{r_m^2}{E\,I}\left[A\,(\phi_1 - \sin\phi_1) - r_m\left(\sin\phi_1 - \frac{\phi_1}{2} - \frac{1}{4}\sin 2\,\phi_1\right)\right]$$

$$+ \frac{r_m}{E\,t}\,[1.94\,\phi_1 - 0.47\sin 2\,\phi_1] \qquad (13)$$

IV.—Check on Cain's Thrusts and Moments

The crown section must be pulled over a distance of $e \sin \phi_1$ (see Fig. 2), in order to bring it back to the center line. As the force, $X = 1$, moves it a distance, y, the total force necessary is,

$$X = \frac{e \sin \phi_1}{y} \qquad (14)$$

The total compressive force on any radial section of the arch due to the water pressure is $p_a\,r_e = p_m\,r_m$ and, therefore, the resulting thrusts are:

At the crown,

At the abutments,

$$\left.\begin{aligned} P_0 &= -p_m r_m + X \\ P_1 &= -p_m r_m + X \cos \phi_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

The moments are:

At the crown,

At the abutments,

$$\left.\begin{aligned} M_0 &= X (A - r_m) \\ M_1 &= X (A - r_m \cos \phi_1) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Comparing Equations (15) and (16) with Professor Cain's equations, it will be seen that the writer's X is equivalent to Professor Cain's $(p\,r - P_0)$. In order to check this, substitute e from Equation (2) and y from Equation (13) in Equation (14). This gives Professor Cain's Equation (10),* with D_s taken from his paper† instead of D, since shear is included. The two methods lead, therefore, to identical results, as they should. This affords a check on the writer's method, as Professor Cain's formulas have already been accepted as correct.

V.—Derivation of More Accurate Formulas

Professor Cain's Equation (6) (Equation (4) of this paper) neglects the rotation brought about by the uniformly distributed normal stresses produced by a force, X, acting at the crown section; he also assumes that the neutral axis coincides with the gravity axis, which is true for straight beams. The deformation brought about by a uniformly distributed normal stress is shown in Fig. 5, where Section ADB is brought to Section $A'D'B'$. The section remains radial, since the elongation of each fiber is proportional to its distance from the center, C, of the arch. The deformation, therefore, consists of an elongation, $D\,D'$, measured on the neutral axis, and a rotation, $\Delta\,d\,\phi'$.

In the preceding treatment, the stress on the radial section due to X was conceived as composed of: (1) A uniform tensile stress, with a resultant, $X \cos \phi$, acting at the center of gravity of the section, a distance, $r_m = 0.5\,(r_e + r_i)$, from the center, C; (2) a moment, $M = X\,(A - r_m \cos \phi)$; and, (3) a shear equal to $X \sin \phi$.

The uniform tensile stress produces rotation, as shown in Fig. 5. The most convenient manner of taking this rotation into account is to express the stresses produced by X', as: (*a*) A tensile stress with a resultant, $X' \cos \phi$, acting at the neutral axis a distance, r_n, from the center, C; (*b*) a moment, $M' = X'\,(A' - r_n \cos \phi)$; and, (*c*) a shear equal to $X' \sin \phi$.

The tensile stress is now no longer uniformly distributed, but the elongations of the various fibers are independent of r, the distance from the center, C (see Equation (25)), so that the rotation is produced by M' alone, and A' is determined from the fact that the rotation at the crown is zero, that is:

$$\int_0^{\phi_1} M'\,d\,\phi = X' \int_0^{\phi_1} (A' - r_n \cos \phi)\,d\,\phi = 0 \quad \ldots\ldots\ldots (17)$$

* *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 237.
† *Loc. cit.*, p. 269.

or,

$$A' = r_n \frac{\sin \phi_1}{\phi_1} = A \frac{r_n}{r_m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

This assumes that E and I are constants.

VI.—Determination of the Force X'

In order to determine the displacement, y', the reasoning given in Article III is followed, but Equation (7) now becomes,

$$d\theta = \frac{M'}{E\,I} r_n\, d\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

in which*,

$$I = r_n\, t\, c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

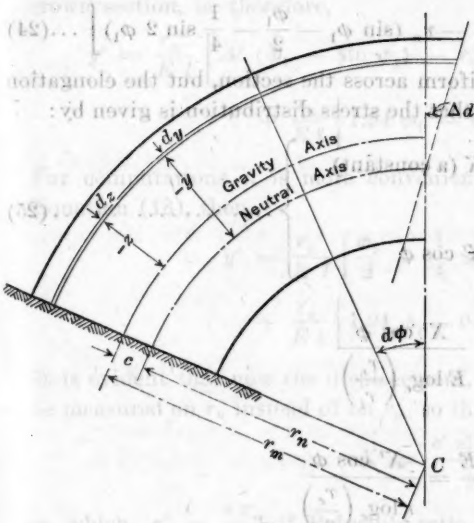

Fig. 6.

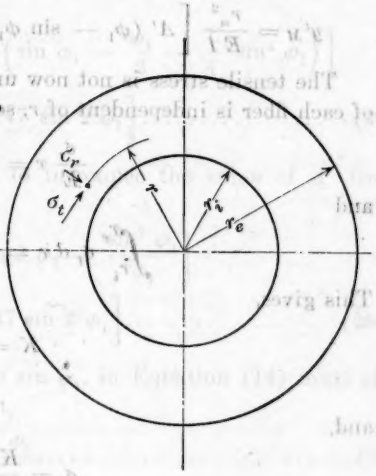

Fig. 7.

is the moment of inertia with respect to the neutral axis, and (see Fig. 6),

$$r_n = r_m - c\,; \qquad c = \frac{k}{1+k} r_m\,; \qquad k = -\frac{1}{t} \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{z}{r_n - z}\, dz \quad \ldots (21)$$

For a rectangular section,

$$k = -1 + \frac{r_m}{t} \log_e \left(\frac{r_e}{r_i}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

in which, $\log_e$ is the natural logarithm.

The difference between I' and $I\left(=\frac{t^3}{12}\right)$ is always small. For $\frac{t}{r_m} = 0.3$ and $r_m = 100$, $k = 0.0076$, $c = 0.75427$, and $I' = 2\,245.74$, while $I = 2\,250$, or

* Bach's "Elasticität und Festigkeit", 9th Edition, p. 502; see, also, Ostenfeld's "Teknisk Elasticitetslaere" (Copenhagen), 3d Edition, p. 472, and Morley's "Strength of Materials".

a difference of only 0.19 per cent. For $\frac{t}{r_m} = 0.6$ and $r_m = 100$, $k = 0.031717$, $c = 3.0742$, and $I' = 17\,878.16$, while $I = 18\,000$, or a difference of only about 0.7 per cent. Consequently, I may always be substituted for I'.

Then, from Equation (20):

$$c = \frac{I}{t\, r_n} = \frac{t^2}{12\, r_n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

This is solved by first assuming $r_n = r_m$ and then substituting the value found for r_n and solving again.

The displacement, y', of the crown section in the direction of X', produced by the bending moment, M', due to $X' = 1$, may now be obtained exactly as in Article III by substituting in Equations (5), (7), (8), and (9), M' for M, A' for A, and r_n for r_m. Then:

$$y'_M = \frac{r_n^{\,2}}{E\, I}\left[A'\,(\phi_1 - \sin\phi_1) - r_n\left(\sin\phi_1 - \frac{\phi_1}{2} - \frac{1}{4}\sin 2\,\phi_1\right)\right] \ldots (24)$$

The tensile stress is not now uniform across the section, but the elongation of each fiber is independent of r, so that the stress distribution is given by:

$$\left.\begin{aligned} &\frac{\sigma_t}{E}\, r = K \text{ (a constant)} \\ \text{and} \\ &\int_{r_i}^{r_e} \sigma_t\, d\, r = X' \cos\phi \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (25)$$

This gives,

$$K = \frac{X' \cos\phi}{E \log_e\left(\frac{r_e}{r_i}\right)}$$

and,

$$\sigma_t = \frac{K\, E}{r} = \frac{X' \cos\phi}{r \log_e\left(\frac{r_e}{r_i}\right)}$$

From Equations (21) and (22),

$$c = \frac{k}{1+k}\, r_m = \frac{-1 + \frac{r_m}{t}\log_e\left(\frac{r_e}{r_i}\right)}{\frac{r_m}{t}\log_e\left(\frac{r_e}{r_i}\right)}\, r_m = r_m - r_n$$

or,

$$\log_e\left(\frac{r_e}{r_i}\right) = \frac{t}{r_n}$$

Introducing this into the equation for σ_t, there results,

$$\sigma_t = \frac{X' \cos\phi}{r\, t}\, r_n \text{ lb. per sq. ft.} \quad \ldots\ldots\ldots\ldots\ldots (26)$$

This stress distribution amounts to a resultant, $X' \cos\phi$, acting in the neutral axis, as it should, since, taking moments about the center, C,

$$\int_{r_i}^{r_o} \sigma_t \, r \, d\,r = \int_{r_i}^{r_o} \frac{X' \cos \phi}{r\,t} \, r_n \, r \, d\,r = X' \, r_n \cos \phi$$

According to Equation (26) the stress at the neutral axis is $\frac{X' \cos \phi}{t}$, or exactly the same value as was used in Article III in deriving Equation (10). Therefore, y'_N, measured on the neutral axis, is,

$$y'_N = \frac{r_n}{2\,E\,t}\left(\phi_1 + \frac{1}{2}\sin 2\,\phi_1\right) \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

And for the shear,

$$y'_S = 2.88\,\frac{r_n}{2\,E\,t}\left(\phi_1 - \frac{1}{2}\sin 2\,\phi_1\right) \ldots\ldots\ldots\ldots\ldots (28)^*$$

The total displacement of the intersection of the neutral axis with the crown section, is, therefore,

$$y' = \frac{r_n^{\,2}}{E\,I}\left[A'\,(\phi_1 - \sin\phi_1) - r_n\left(\sin\phi_1 - \frac{\phi_1}{2} - \frac{1}{4}\sin^2\phi_1\right)\right] + \frac{r_n}{E\,t}\left[1.94\,\phi_1 - 0.47\,\sin 2\,\phi_1\right] \ldots\ldots\ldots\ldots\ldots (29)$$

For computations it is more convenient to introduce the value of A' from Equation (18), then,

$$y' = \frac{r_n^{\,3}}{E\,I}\left[\frac{\phi_1}{2} + \frac{1}{4}\sin 2\,\phi_1 - \frac{\sin^2\phi_1}{\phi_1}\right] + \frac{r_n}{E\,t}\left[1.94\,\phi_1 - 0.47\,\sin 2\,\phi_1\right] \ldots\ldots\ldots\ldots\ldots (29a)$$

It is evident that now the displacement, $e \sin \phi_1$, in Equation (14) must also be measured on r_n instead of on r_m, so that,

$$X' = \frac{e' \sin \phi_1}{y'} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

in which, $e' = \frac{e\,r_n}{r_m}$, if Poisson's ratio is assumed to be zero. The correct value for e' will be given in Article VIII.

VII.—Stresses Due to the Force X'

To determine the stresses produced by X' acting at the crown section, it is more convenient to locate the two equal but opposite auxiliary forces, X', at r_m, so that the resulting tensile stress is uniformly distributed and equal to

*Professor Cain has called the writer's attention to the fact that r_m and not r_n should be used in this equation and also has pointed out that the error was most likely negligible. Using r_n simplifies the calculations (see Equation (56)). Moreover the coefficient, $2.88 = 1.2 \times 2.4$, is not exact. The value 2.4, has already been discussed in connection with Equation (12); the factor, 1.2, is derived on the assumption that the shear is distributed according to a parabola and is derived for a straight beam and not for an arch. Also, this coefficient, 1.2, is an average value found by the work formula and not an exact value.

The actual value of the coefficient, 2.88, as determined by the work formula for the arch considered in Article XII, for which, $\frac{t}{r_m} = 0.5$, is 2.63 for Poisson's ratio, $m = 8$, and 2.63 r_m is in this case 2.687 r_n. Therefore, 2.88 r_n in Equation (28) is already too large and if $m = 8$, it is nearer the correct value, in this case, than 2.88 r_m.

$X' \frac{\cos \phi}{t}$. The moment must then also be referred to r_m and is $M = X'(A' - r_m \cos \phi)$. The stresses due to the bending moment alone are,

$$\sigma_t = \frac{M}{I} \frac{y}{1 - \frac{y}{r_n}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

y is measured from the neutral axis (see Fig. 6). For the extrados and intrados, respectively,

$$\left.\begin{aligned} y_e &= -\left(\frac{t}{2} + c\right) \\ y_i &= \frac{t}{2} - c \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32)$$

The bending stresses at the extrados and intrados, respectively, are then,

$$\left.\begin{aligned} \sigma_{te} &= k_e \frac{M}{t^2}, \text{ in which, } k_e = \frac{r_n}{r_e}\left(\frac{1}{2} + \frac{c}{t}\right) \\ \sigma_{ti} &= k_i \frac{M}{t^2}, \text{ in which, } k_i = \frac{r_n}{r_i}\left(\frac{1}{2} - \frac{c}{t}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (33)$$

In Equation (1), $k_e = k_i = 6$. For large values of $\frac{t}{r_m}$, this may lead to a considerable error.

The stresses produced by X' on a radial section ϕ degrees from the crown section, are, therefore, for the extrados and intrados, respectively,

$$\left.\begin{aligned} \sigma_{te}' &= X'\left[k_e \frac{A' - r_m \cos \phi}{t^2} + \frac{\cos \phi}{t}\right] \text{ lb. per sq. ft.} \\ \sigma_{ti}' &= -X'\left[k_i \frac{A' - r_m \cos \phi}{t^2} - \frac{\cos \phi}{t}\right] \text{ lb. per sq. ft.} \end{aligned}\right\} \quad \ldots\ldots\ldots (34)$$

To these stresses must be added those produced by the water pressure, p_a, acting on the extrados. If the ordinary cylinder formula is used, this is,

$$\sigma_t'' = -\frac{p_a}{t} r_e \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35)$$

VIII.—Stresses Due to Water Pressure

The assumption that Poisson's ratio is zero is not correct for thick cylinders. The stresses and deformations for a thick hollow cylinder, are* (see Fig. 7):

1.—The tangential stress,

$$\sigma_t'' = -\left(1 + \frac{r_i^2}{r^2}\right) \frac{r_e^2}{r_e^2 - r_i^2} p_a$$

or for the extrados and intrados, respectively,

$$\left.\begin{aligned} \sigma_{te}'' &= -\left(\frac{r_m}{t} + \frac{1}{4}\frac{t}{r_m}\right) p_a \\ \text{and} \quad \sigma_{ti}'' &= -\left(1 + \frac{r_m}{t} + \frac{1}{4}\frac{t}{r_m}\right) p_a \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (36)$$

* Bach's "Elasticität und Festigkeit," 9th Edition, Paragraph 58; see, also, Merriman's "Mechanics".

2.—The radial stress,

$$\sigma_r'' = -\left(1 - \frac{r_i^2}{r^2}\right) \frac{r_e^2}{r_e^2 - r_i^2} p_a \quad \ldots\ldots\ldots\ldots\ldots\ldots (37)$$

(Here, p_a is that part of the water pressure which is supported by the arch. As will be seen, both the tangential and the radial stresses are independent of the axial stress, σ_x, as they should be for a cylinder. That is not true for an arch dam, for the reason that the vertical stress, σ_x, influences the deformation.)

3.—The tangential deformation, ϵ'. If σ_x is the vertical stress produced by the weight of the concrete above the arch under consideration and by the water acting vertically on the up-stream face, the tangential deformation per unit length a distance, r_n ft. from the center, is:

$$\epsilon' = -\frac{1}{E}\left[\frac{m-1}{m} + \frac{m+1}{m}\frac{r_i^2}{r_n^2}\right]\frac{r_e^2}{r_e^2 - r_i^2} p_a - \frac{\sigma_x}{m E} \quad \ldots\ldots (38)$$

For slide-rule computation this can be more conveniently written,

$$\epsilon' = -\frac{1}{E}\left[\frac{m-1}{m} + \frac{m+1}{m}\left(\frac{1 - \frac{t}{2\,r_m}}{1 - \frac{c}{r_m}}\right)^2\right] \frac{\left(1 + \frac{t}{2\,r_m}\right)^2}{\frac{2\,t}{r_m}} p_a - \frac{\sigma_x}{m E} \quad \ldots\ldots\ldots\ldots (38a)$$

The displacement, e', measured at the neutral axis, is therefore,

$$e' = -r_n \epsilon' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39)$$

If Poisson's ratio is assumed to be zero, that is, if the cylinder formula is used, the deformation is,

$$\epsilon'' = -\frac{p_a r_e}{E t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38b)$$

For $\frac{t}{r_m} = 0.5$, this is only 4.3% higher than the exact value, computed for $m = 8$; but sometimes a rather small change of X' may produce a considerable change in the resulting stresses.

When applying these formulas to an arch dam, it is tacitly assumed that the arches are free to deform radially but this is not true at the abutments. In deriving Equations (36) to (38), inclusive, σ_x has been assumed as constant, that is, independent of r. This is, of course, not correct, but the average value of σ_x is practically the same whether the reservoir is empty or full. To take account of the variation of σ_x due to cantilever action would lead to great complications, and for the bottom part of the dam, the assumption that E and σ_x are constants, produces two errors, which are of opposite sign (see Article XVI).

It has further been assumed that the arch completely fills the site, so that if there were no concrete above the arch, both σ_x and ϵ' would be zero with the reservoir empty; or, in other words, it was assumed that no shrinkage would

take place. This assumption may possibly be approximately correct for the very thick bottom part of a great arch dam, especially if σ_x is assumed as somewhat less than the average value for the vertical stress. Where the assumption is acceptable and p_a is the only water pressure acting on the extrados, Equations (38) and (39) give the displacement which X' must annul, and Equation (36) gives the tangential stress which must be added to Equation (34) to obtain the resulting stresses.

For the greater part of the dam, the shrinkage of the concrete must be accepted as a fact. To counteract this shrinkage, contraction joints should be provided at suitable intervals and after the dam has set and the chemical heat has been dissipated, pressure grouting should be effected.

With the reservoir empty, that is, for $p_a = 0$, there is a uniform tangential compression σ_{tg} in the arch and a variable vertical compression, σ_x, so that any element of the arch has undergone a tangential deformation.

$$\varepsilon_t = -\frac{\sigma_x}{m\,E} + \frac{\sigma_{tg}}{E} \qquad (40)$$

Both σ_x and σ_{tg} are compressions and, therefore, negative. Hence, when the water pressure, p_a, acts on the extrados, the deformation, ϵ', is computed from Equation (38), with $\sigma_x = 0$, and the tangential stress results from adding σ_{tg} to Equation (36) and then adding Equation (34) to obtain the resulting stress.

IX.—Accuracy of Assumptions

It has been assumed that E is constant and that the normal (radial) planes remain plane. Both these assumptions are at least approximately correct when the stresses are moderate, and both contribute an error on the side of safety. Professor Bach believes that radial planes will remain at least approximately plane also for such material where E is not constant,* and this will probably still be approximately true even when the abutment does not lie in one radial plane. For high stresses, E cannot be assumed as constant and the stress distribution will no doubt be materially changed. This will be discussed in detail in Article XVII.

The division of the water load between the cantilever and the arch cannot be made accurately and, moreover, it is not likely that p_a is constant for any one arch or even symmetrical with respect to the crown section. The influence of p_a decreases rapidly toward the abutment, and a small variation in symmetry should not affect the resulting stress greatly, so that in general the crown value of p_a may be used as a fair approximation, especially if care is taken to select p_a somewhat larger than for the determination of the arch stresses.

Some objection may be raised to the rather general application of Poisson's ratio. Tests made on hollow steel cylinders agree quite closely with Lamé's theory.† The writer does not know of any tests having been made by subjecting hollow concrete cylinders to external pressure, but Professor Bach

* "Elasticität und Festigkeit," 9th Edition, p. 289, last footnote.

† See, for example, Fig. 333(*b*), p. 350, of Goodman's "Mechanics Applied to Engineering," 1912 Edition.

reports some tests to destruction on hollow unreinforced concrete cylinders by subjecting them to internal pressure.* The internal diameter was 70 cm., and the thickness, 8, 12, and 18 cm. The tangential stress as computed by the Lamé formula was 15.7 kg. per sq. cm. and the tensile strength of the concrete was found to be 15.5 kg. per sq. cm. If the stress had been uniformly distributed throughout the largest cylinder (18 cm. thick) it should have withstood a pressure 30% higher. These tests show, therefore, an excellent agreement with the Lamé formula. It is likely, however, that cylinders tested under external loading would show some deviation, because E is not nearly so constant in compression as in tension.

If it were feasible properly to take account of the swelling of the up-stream face due to water-soaking, to the time effect (the yielding of the concrete with time in proportion to the stress), and to the restraining action of the foundation as distinguished from the abutment, a still more advantageous stress distribution probably would be found.

X.—Radial Deflection of Arch

The displacement, BF, of the point, B (Fig. 2) is $e\ (1 - \sin \phi_1)$ in the direction of the center line. This displacement is due to the water pressure on the up-stream face and to the fact that the arch has been slid back from A' to A. To this displacement must be added the displacement in the direction of the center line produced by the force, X', found previously.

The displacement caused by $X' = 1$ is most conveniently determined by referring the normal stresses and the moments to r_n as in Articles V and VI. The rotation produced is given by Equation (19) and the corresponding displacement of the crown section in the direction of the center line, BC (taken positive from B toward C), neglecting the change in thickness of the arch (see Fig. 3), is

$$\eta_M = \int_0^{\phi_1} 2\, r_n \sin \frac{\phi}{2}\,(A' - r_n \cos \phi)\,\frac{r_n}{E\,I} \cos \frac{\phi}{2}\, d\,\phi$$

or, integrating and rearranging,

$$\eta_M = \frac{r_n^2}{E\,I}\left[A'\,(1 - \cos \phi_1) - \frac{r_n}{4}\,(1 - \cos 2\,\phi_1)\right] \ldots\ldots\ldots (41)$$

Since according to Equation (26) the tension at the neutral axis is $\frac{X' \cos \phi}{t}$ (see, also, Fig. 4), the displacements due to tension and shear produced by $X' = 1$, measured on the neutral axis, are,

$$\eta_N = -\int_0^{\phi_1} \frac{\cos \phi}{E\,t}\, r_n \sin \phi\, d\,\phi = -\frac{r_n}{4\,E\,t}\,(1 - \cos 2\,\phi_1) \ldots\ldots (42)$$

and,

$$\eta_S = 2.88 \int_0^{\phi_1} \frac{\sin \phi}{E\,t}\, r_n \cos \phi\, d\,\phi = 2.88\,\frac{r_n}{4\,E\,t}\,(1 - \cos 2\,\phi_1) \ldots\ldots (43)$$

The deflection caused by $X' = 1$, is, therefore,

$$\eta = \eta_M + \eta_N + \eta_S$$

* "Elasticität und Festigkeit," 9th Edition, p. 579, footnote.

or,

$$\eta = \frac{r_n^2}{E\,I}\left[A'\,(1-\cos\phi_1) - \frac{r_n}{4}\,(1-\cos 2\,\phi_1)\right] + 0.47\,\frac{r_n}{E\,t}\,(1-\cos 2\,\phi_1) \quad \ldots\ldots(44)$$

The total deflection will be,

$$\eta_i = \eta\,X' - \epsilon'\,r_n\,(1-\cos\phi_1) \quad \ldots\ldots(45)$$

Here, ϵ' is, as before, determined from Equation (38). If the deflection for the up-stream face is wanted rather than for the neutral axis, the change of thickness of the arch, which may be found from Equation (37), must be considered. This will generally and perhaps always be negligible. If pressure grouting is undertaken, this may also affect the deflection slightly.

XI.—Temperature Variations, Swelling, and Shrinkage.

Let u_e and u_i be the deformations per unit length at the extrados and intrados, respectively, caused by the temperature variation, swelling, and shrinkage. Let u_e and u_i be positive when they represent increase in length and assume that u varies from u_e to u_i in such a manner that radial vertical planes remain plane.

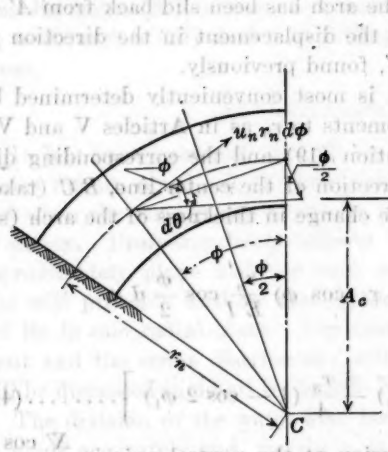

Fig. 8.

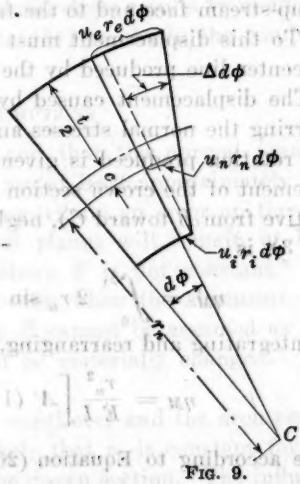

Fig. 9.

If in Fig. 8 the semi-arch is free to move at the crown, but is fixed at the abutment, the deformations, u, produce a rotation of the crown section and also displacements parallel to and at right angles to X_c. The force, X_c, acting a distance, A_c, from the center, C, of the arch, and considered positive when in tension, is the force necessary to bring the crown section back into the center line, $C\,D$. In order to determine X_c and A_c it is necessary, therefore, to determine the rotation and the displacement in the direction of X_c of the crown section.

The unit deformation of the neutral axis (see Fig. 9) is determined by,

$$u_n\,r_n = u_i\,r_i + \frac{u_e\,r_e - u_i\,r_i}{t}\left(\frac{t}{2} - c\right)$$

which, for $c = 0$, gives,

$$u_n = \frac{u_e r_e + u_i r_i}{2 r_n} \quad \ldots\ldots (46)$$

The rotation is,

$$d \Delta \phi = d \theta = \frac{u_e r_e - u_i r_i}{t} d \phi \quad \ldots\ldots (47)$$

The rotation of the crown section is, therefore,

$$\theta = (u_e r_e - u_i r_i) \frac{\phi_1}{t} \quad \ldots\ldots (48)$$

The displacement of the crown section in the direction of X_c is (see Fig. 8),

$$y_c = \int_0^{\phi_1} 2 r_n \sin^2 \left(\frac{\phi}{2}\right) d \theta + \int_0^{\phi_1} u_n r_n \cos \phi \, d \phi$$

Integrating,

$$y_c = \frac{r_n}{t} (u_e r_e - u_i r_i) (\phi_1 - \sin \phi_1) + \tfrac{1}{2} (u_e r_e + u_i r_i) \sin \phi_1 \quad \ldots\ldots (49)$$

The force, X_c, acting at a distance, A_c, from the center, C, must annul both the rotation, θ, and the displacement, y_c. From Equation (19),

$$- \theta = \int_0^{\phi_1} \frac{M_c}{E I} r_n \, d \phi$$

in which,

$$M_c = X_c (A_c - r_n \cos \phi)$$

so that,

$$\theta = - \frac{X_c r_n}{E I} (A_c \phi_1 - r_n \sin \phi_1) \quad \ldots\ldots (50)$$

and θ is given by Equation (48). The displacement produced by X_c is obtained from Equation (29) by substituting A_c for A' and multiplying by X_c; it is:

$$y_c = - X_c \left\{ \frac{r_n^2}{E I} \left[A_c (\phi_1 - \sin \phi_1) - r_n \left(\sin \phi_1 - \frac{\phi_1}{2} - \frac{1}{4} \sin 2 \phi_1 \right) \right] + \frac{r_n}{E t} [1.94 \phi_1 - 0.47 \sin 2 \phi_1] \right\} \quad \ldots\ldots (51)$$

In addition, y_c is given by Equation (49). Solving Equations (50) and (51) gives X_c and A_c. The deflection of the crown section in an up-stream and down-stream direction caused by the deformations, u, may be obtained from Equation (44) by substituting A_c for A and multiplying by X_c, giving,

$$\eta_c = X_c \left\{ \frac{r_n^2}{E I} \left[A_c (1 - \cos \phi_1) - \frac{r_n}{4} (1 - \cos 2 \phi_1) \right] + 0.47 \frac{r_n}{E t} (1 - \cos 2 \phi_1) \right\} \quad \ldots\ldots (52)$$

The stresses produced by X_c acting a distance, A_c, from the center is found from Equation (34) by substituting X_c and A_c for X' and A'. The final resulting deformations and stresses are then found by simple addition.

The computations are greatly simplified, if θ in Equation (48) is zero, that is,

$$u_c r_c = u_i r_i = u_n r_n \quad \ldots\ldots (53)$$

Then, from Equation (50):

$$A_c = r_n \frac{\sin \phi_1}{\phi_1} = A' \quad \ldots\ldots (54)$$

In this case, u'_n, the deformation per unit length at the neutral axis, due to temperature variation, swelling, and shrinkage, may be added directly to ϵ' of Equation (38) and the force required, including temperature variation, etc., may be determined from:

$$X_c' = \frac{\epsilon' + u_n'}{y'} \sin \phi_1 \quad \ldots\ldots (55)$$

in which, ϵ' is the deformation at the neutral axis per unit length due to the water pressure acting on the extrados. In case $\epsilon' + u'_n = 0$, $X'_c = 0$, and the only tangential stresses are those given by Equation (36); there are no bending stresses. This is also evident from an inspection of Fig. 2 and from the reasoning of Article II.

Obviously, good construction demands that the pressure grouting should take place while the concrete is at or near its annual minimum temperature. Also, near the foundation, where pressure grouting cannot be made effective, the concrete should be poured during the coldest part of the season, or preferably during that period when the air temperature is approximately equal to or somewhat below the annual mean. In this manner the temperature variations will lower the maximum stresses, since ϵ' and u'_n are of opposite sign.

XII.—Numerical Computations

In order to facilitate the use of the formulas here presented and to enable the computations to be made by slide-rule, some of the equations can be put in a form that facilitates the use of tables and curves. Let,

$$B = 12 \frac{r_n^2}{t^2} \left[\frac{\phi_1}{2} + \frac{1}{4} \sin 2\phi_1 - \frac{\sin^2 \phi_1}{\phi_1} \right] + 1.94 \phi_1 - 0.47 \sin 2\phi_1 . \quad (56)$$

so that, from Equation (29a),

$$y' = \frac{r_n}{E\,t} B \quad \ldots\ldots (57)$$

The value of B is given in Table 1 and is plotted in Fig. 10. The values of $100 \frac{c}{r_m}$ (c when $r_m = 100$ ft.), k_c and k_i are given in Table 2 and plotted in Figs. 11 and 12.

From Equations (30) and (57),

$$X' = -\frac{\epsilon'}{B \frac{r_n}{t}} E \sin \phi_1 = -\frac{\epsilon'}{B} t E \sin \phi_1 \quad \ldots\ldots (58)$$

or,

$$\frac{X'}{t} = -\frac{\epsilon'}{B} E \sin \phi_1$$

If this value and the value for A' are introduced into Equation (34) and the equation is re-arranged, there results,

$$\sigma_{te}' = -\frac{\varepsilon'}{B} E \sin \phi_1 \left\{ k_e \frac{r_m}{t} \left[\frac{\sin \phi_1}{\phi_1} - \cos \phi - \frac{c}{r_m} \frac{\sin \phi_1}{\phi_1} \right] + \cos \phi \right\}$$

$$\sigma_{ti}' = \frac{\varepsilon'}{B} E \sin \phi_1 \left\{ k_i \frac{r_m}{t} \left[\frac{\sin \phi_1}{\phi_1} - \cos \phi - \frac{c}{r_m} \frac{\sin \phi_1}{\phi_1} \right] - \cos \phi \right\} \quad \text{.(59)}$$

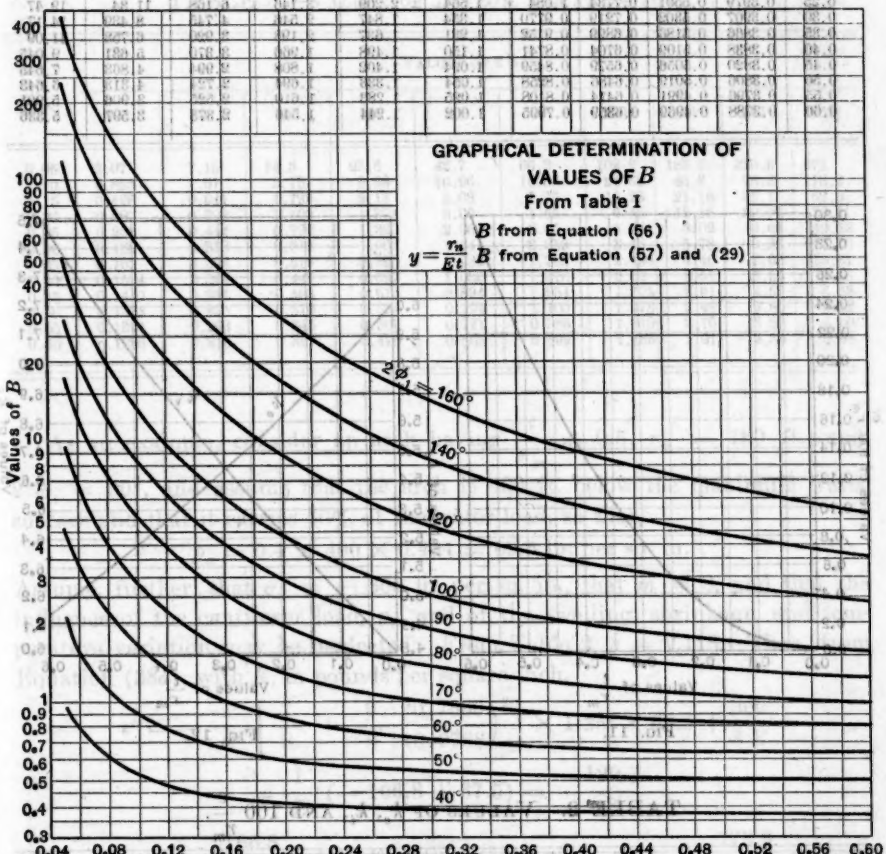

FIG. 10.

At the crown, $\cos \phi = 1$, and, at the abutment, it is $\cos \phi_1$. The value of $\frac{\sin \phi_1}{\phi_1}$ can be determined quite closely on the slide-rule by first computing

$$1 - \frac{\sin \phi_1}{\phi_1} = \phi_1 - \frac{\sin \phi_1}{\phi_1} \quad \text{.....(}$$

TABLE 1.—VALUES OF B FROM EQUATION (56).

$\frac{t}{r_m}$	VALUES OF $2\phi_1$.									
	40°	50°	60°	70°	80°	90°	100°	120°	140°	160°
0.05	0.9511	2.094	4.665	9.335	17.88	30.25	49.64	116.4	236.4	429.9
0.10	0.5188	0.8876	1.619	2.887	5.131	8.338	13.31	30.26	60.53	109.2
0.15	0.4388	0.6644	1.069	1.695	2.772	4.285	6.588	14.33	28.01	49.85
0.20	0.4108	0.5862	0.8605	1.277	1.885	2.866	4.235	8.748	16.61	29.07
0.25	0.3979	0.5501	0.7734	1.084	1.564	2.209	3.146	6.168	11.84	19.47
0.30	0.3907	0.5302	0.7219	0.9770	1.354	1.847	2.546	4.745	8.439	14.17
0.35	0.3866	0.5186	0.6899	0.9152	1.231	1.637	2.198	3.920	6.753	11.09
0.40	0.3838	0.5109	0.6704	0.8741	1.150	1.498	1.966	3.370	5.631	9.045
0.45	0.3820	0.5056	0.6579	0.8459	1.094	1.402	1.808	2.994	4.863	7.645
0.50	0.3806	0.5019	0.6486	0.8258	1.054	1.333	1.694	2.724	4.313	6.643
0.55	0.3796	0.4991	0.6414	0.8108	1.025	1.283	1.610	2.525	3.906	5.901
0.60	0.3788	0.4969	0.6359	0.7995	1.002	1.244	1.546	2.373	3.597	5.336

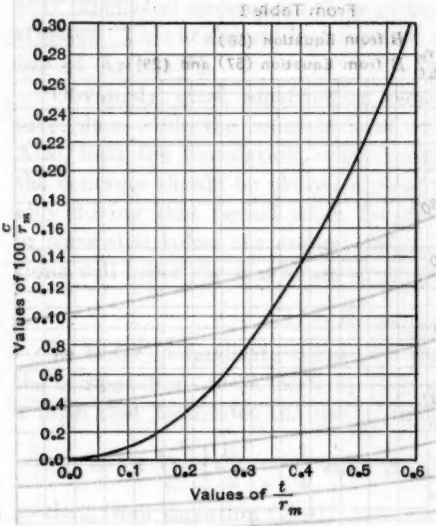

FIG. 11.

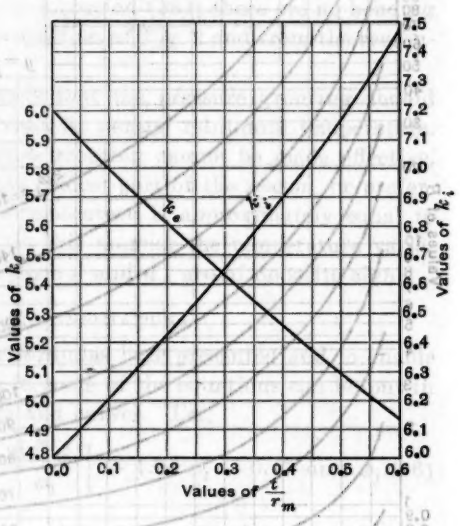

FIG. 12.

TABLE 2.—VALUES OF k_e, k_i, AND $100\frac{c}{r_m}$.

$\frac{t}{r_m}$ =	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60
k_e =	5.900	5.801	5.710	5.618	5.528	5.438	5.351	5.264	5.181	5.096	5.013	4.932
k_i =	6.105	6.210	6.312	6.422	6.535	6.654	6.775	6.902	7.030	7.167	7.310	7.457
$100\frac{c}{r_m}$ =	0.000	0.084	0.188	0.334	0.525	0.754	1.030	1.347	1.712	2.118	2.572	3.074

The radial deflection at the crown due to $X' = 1$ is, from Equation (44),

$$\eta = \frac{r_n}{E\,t}\,D \qquad \ldots\ldots\ldots\

in which,

$$D = 3\frac{r_m^2}{t^2}\left[2\,\frac{2\sin\phi_1 - \sin 2\phi_1}{\phi_1} - (1 - \cos 2\phi_1)\right] + 0.47\,(1 - \cos 2\phi_1) \quad \ldots\ldots(61)$$

The value for D is given in Table 3 and plotted in Fig. 13. The total deflection is then given by Equation (45).

TABLE 3.—VALUES OF D FROM EQUATION (61).

$\frac{t}{r_m}$	VALUES OF $2\phi_1$.								
	40°	50°	60°	70°	80°	90°	100°	120°	140°	160°
0.05	2.97	7.15	14.3	25.8	42.7	66.2	102.2	185.5	310.8	471
0.10	0.823	1.91	3.75	6.68	10.96	16.58	24.68	46.8	78.2	118.2
0.15	0.426	0.941	1.793	3.18	5.08	7.75	11.26	21.16	35.1	52.9
0.20	0.287	0.602	1.109	1.894	3.02	4.55	6.56	12.18	20.07	30.1
0.25	0.223	0.444	0.792	1.320	2.066	3.07	4.38	8.02	13.01	19.52
0.30	0.188	0.358	0.618	1.004	1.541	2.259	3.18	5.78	9.26	13.70
0.35	0.167	0.308	0.516	0.820	1.235	1.785	2.486	4.40	7.03	10.31
0.40	0.1534	0.274	0.449	0.697	1.032	1.470	2.02	3.51	5.54	8.06
0.45	0.1440	0.251	0.408	0.614	0.894	1.254	1.705	2.91	4.53	6.53
0.50	0.1373	0.235	0.370	0.554	0.794	1.100	1.478	2.48	3.80	5.42
0.55	0.1324	0.223	0.345	0.507	0.721	0.986	1.310	2.16	3.26	4.60
0.60	0.1286	0.213	0.327	0.476	0.665	0.899	1.183	1.911	2.85	3.98

As an example, consider an arch having $\frac{t}{r_m} = 0.5$, $r_m = 100$ ft., and $2\phi_1 = 90°$, and assume that the arch is 400 ft. below the maximum water surface and that it carries 40% of the water load, so that,

$$p_a = 0.4 \times 400 \times 0.434 = 69.5 \text{ lb. per sq. in.}$$

Assume, further, that $\sigma_x = -300$ lb. per sq. in., that $m = 8$, and that the influence of the cantilever load, p_c, and of the swelling, shrinkage, and temperature variation may be neglected. From Table 2, $c = 2.118$. Then, from Equation (38a), with E, in pounds per square inch,

$$\varepsilon' = -\frac{1}{E}\left[\frac{7}{8} + \frac{9}{8}\left(\frac{0.7500}{0.9788}\right)^2\right] \times 1.25^2 \times 69.5 + \frac{300}{8E}$$

$$= \frac{1}{E}(-166.8 + 37.5) = -\frac{129.3}{E}$$

If $\delta_x = 0$, $\varepsilon' = -\frac{166.8}{E}$, while Equation (38$b$) gives $\varepsilon'' = -\frac{173.9}{E}$, or only 4.26% greater than ε' for $\sigma_x = 0$.

From Equation (36) is obtained,

$$\sigma''_{te} = -147.7; \text{ and } \sigma''_{ti} = -217.2$$

From Tables 1 and 2, $B = 1.333$, $k_e = 5.096$, and $k_i = 7.167$, so that from Equation (59), the stress at the crown section is,

$$\sigma_{te}' = \frac{129.3}{1.333}\left\{5.096 \times 2\,[0.90032 - 1.0 - 0.0191] + 1\right\} \times 0.70711$$

$$= -14.4 \text{ lb. per sq. in. (compression).}$$

Likewise it is found that $\sigma'_{ti} = 185.3$ lb. per sq. in. (tension). For the stresses at the abutment, where $\cos \phi = \cos \phi_1 = 0.70711$, there is found $\sigma'_{te} = 170.2$ lb. per sq. in. (tension) and $\sigma'_{ti} = -122.9$ lb. per sq. in. (compression). The resulting tangential stresses are then obtained by adding to these the proper values of σ''_t found previously. The results are given in Table 4.

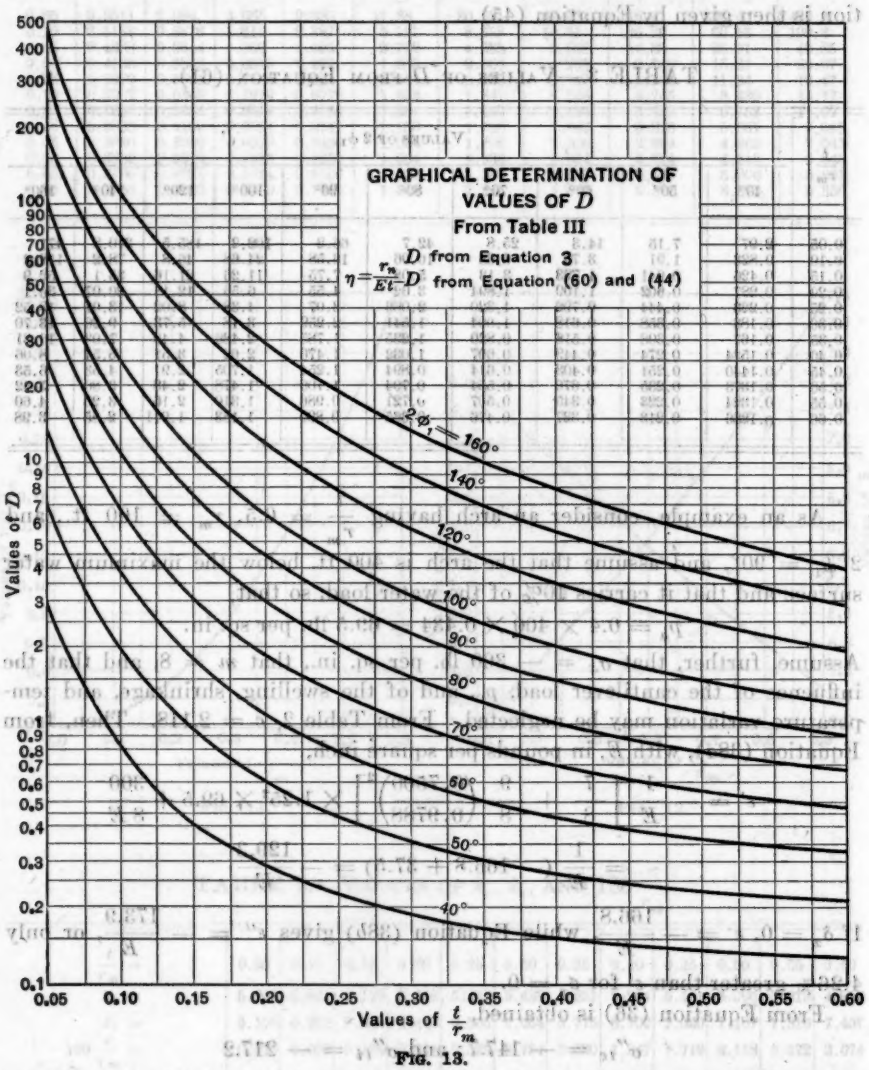

FIG. 13.

For $\sigma_x = -300$ lb. per sq. in. (compression), $X' = \frac{129.3}{1.333} 0.707 \times 50 =$ 3 430 lb.,* while for $\sigma_x = 0$, $X' = 4\,420$. Professor Cain's formulas assume

* When computing stresses in pounds per square inch, it is more convenient to express X' in pounds acting on an arch slice $\frac{1}{12}$ in. thick, as has been done here, but the formulas assume X' acting upon an arch slice 1 ft. thick.

$\sigma_x = 0$ and his value for D_s was computed as $0.05606\ \phi_1$, giving $p_t - P_o = 4\,565.8$ lb., or only about 3.3% greater than the writer's value for X'. The moments found at the crown and at the abutments by Professor Cain's formulas are 45 513 and 88 215 as against 52 500 and 77 100 by the writer's formulas when $\sigma_x = 0$; or Professor Cain's values for the moments at the crown and at the abutments are 86.7% and 114% of those found by the writer's formulas. This is a fairly close check considering that the arch is rather heavy. The influence of σ_x is considerable in this case, as will be seen from the computations.

TABLE 4.—COMPARISON OF STRESSES.

	Crown Section.		Abutment Section.	
	Extrados.	Intrados.	Extrados.	Intrados.
Jakobsen's formulas, $\sigma_x = 300$	− 162.1	− 31.9	+ 22.5	− 340.1
Cain's formulas and Equation (1)	− 191.7	+ 26.8	+ 102.5	− 310.9
Jakobsen's formulas, $\sigma_x = 0$	− 166.2	+ 21.3	+ 71.3	− 375.2

The radial deflection is for $\sigma_x = 0$, since $X' = 4\,420$ and $D = 1.100$, so that from Equation (60):

$$\eta = \frac{97.88}{50\,E}\,1.10 = \frac{2.155}{E}$$

and from Equation (45),

$$E\eta_t = 2.155 \times 4\,420 + 166.8 \times 97.88 \times 0.293$$

or,

$$\eta_t = \frac{14\,320}{E}$$

For $\sigma_x = -300$ lb. per sq. in., $\eta_t = \frac{12\,200}{E}$. Professor Cain's formula including shear* gives $\frac{15\,400}{E}$ as the deflection which agrees quite well with the value found by the writer's formula. Neglecting shear, Professor Cain's formula gives $\frac{10\,420}{E}$; for thick arches shear contributes considerably to the deflection.

That the influence of a temperature variation is quite considerable may be shown readily as follows: For $\sigma_x = -300$ lb. per sq. in. (compression), $\varepsilon' = -\frac{129.3}{E}$, and if $\varepsilon' + u_n' = 0$, $X_c' = 0$. The increase in length for an increase of temperature of 1° Fahr. is approximately 6×10^{-6} ft. per ft. of length, and the temperature increase required to annul X_c' is, if $E = 2 \times 10^6$,

$$T = \frac{129.3}{2 \times 6} = 10.8° \text{ Fahr.}$$

* *Transactions*, Am. Soc. C. E., Vol. LXXXV (1915), p. 270.

XIII.—Influence of Cantilever Load and Pressure Grouting

That part, p_c, of the water pressure, which is carried by the cantilever and transmitted vertically by shear, also has an effect on the arch in that it produces a radial stress, σ_r, similar to σ_y in Fig. 1. This radial stress, σ_r, produces a tangential stress, which may be found as follows.

At the up-stream face,*

$$\sigma_r = \sigma_x \cot^2 \alpha + p_a (1 - \cot^2 \alpha)$$

in which, α is the angle which the up-stream face makes with the horizontal. If σ_x and α are both fairly small, σ_{re} is nearly equal to p_a. Assuming the down-stream face to be vertical, as is generally the case with constant-angle arch dams, except quite near the bottom, $\sigma_{ri} = 0$. For a multiple-arch buttress, the writer has shown† that σ_r decreases nearly uniformly; this variation may be assumed to hold at least approximately in this case also.

Professor Cain has shown that for a triangular section of a gravity dam with a nearly vertical up-stream face, σ_r is almost constant for the entire section.‡ In the case of an arch dam that acts also as a gravity dam, no doubt σ_r likewise varies with ϕ. The actual determination of σ_r is complicated, and since the purpose of the present discussion is to show that the influence of σ_r on the stress distribution is quite small, the assumption made seems permissible.

If the assumption that σ_r is constant be made, the result is nearly the same as if the temperature of the concrete were increased sufficiently to produce a unit elongation, $\frac{-\sigma_r}{m E}$, so that Equation (53) holds approximately; this case was discussed in Article XII.

It seems to the writer that no matter how σ_r varies, it cannot influence the stress distribution greatly except for very short and thick arches, which are not likely to be of any great practical importance.

Assuming, therefore, that $\sigma_{re} = -p_a$ and $\sigma_{ri} = 0$ and that σ_r varies linearly, the influence of p_c on the arch deflections and stresses may be found by adding $\frac{p_c}{m E}$ to u_e in Article XI; or if the influence of p_c alone is wanted, by putting $u_e = \frac{p_c}{m E}$ and $u_i = 0$ and determining the corresponding values of X_c, A_c, and the deflection in an up-stream direction.

For the arch considered in Article XII,

$$p_c = 0.6 \times 400 \times 0.434 = 104.1 \text{ lb. per sq. in.}$$

so that

$$u_e = \frac{104.1}{8 E} = \frac{13.02}{E}.$$

From Equation (48),

$$\theta = \frac{25.58}{E}$$

* Equation (30) of the writer's paper "Stresses in Multiple-Arch-Dams," *Transactions*, Am. Soc. C. E., Vol. LXXXVII (1924), p. 298.

† *Loc. cit.*, p. 306.

‡ "Stresses in Masonry Dams," *Transactions*, Am. Soc. C. E., Vol. LXIV (1909), p. 208.

From Equation (49),

$$y_c = \frac{624.8}{E}$$

Equations (50) and (51) then give:

$$25.58 = -0.0094\, X_c\, [0.7854\, A_c - 69.2]$$

and,

$$624.8 = -X_c\, [0.072\, A_c - 4.242]$$

Solving,

$$A_c = 107.4 \text{ ft. and } X_c = -180.2 \text{ lb.}$$

The stresses produced at the abutment are, therefore, from Equation (34):

$$\sigma'_{te} = -16.0 \text{ lb. per sq. in (compression)}$$

and,

$$\sigma'_{ti} = +21.5 \text{ lb. per sq. in. (tension).}$$

These stresses tend to equalize those produced by the water pressure. The tension at the abutment is 22.5 lb. per sq. in., so that the resulting tension is only 6.5 lb. per sq. in. The value of $\frac{p_c}{m\,E}$ is 13.0 lb. per sq. in. For a shorter arch the influence of p_c will be greater.

In Article VIII the grouting pressure σ_{tg} was assumed to be uniformly distributed tangential compression. This is true where the contraction joints are fairly close together and all are grouted simultaneously. By suitable grouting methods, however, it is possible to produce in the unloaded arch bending moments that would tend to neutralize those due to the water pressure. At times, this may be an advantage well worth considering. For example, if all joints except the one at the crown section are grouted first, the grouting of the crown joint under pressure tends to put compression in the extrados of the abutment section.

It is customary to assume that at the bottom of an arch dam the cantilever carries the entire load and the arch carries nothing. This may not be true in a high dam, such as the Pacoima Dam, where the bottom arches support nearly 400 ft. of concrete and their thickness may prevent shrinkage to a considerable extent.

XIV.—Formulas for the Stresses Produced by X, when Poisson's Ratio Is Considered

In computing the stresses and displacements caused by a force, X (Fig. 2), Poisson's ratio was assumed to be zero, or, in other words, it was assumed that normal stresses are produced in one direction only, namely, tangentially, and that the normal stresses in radial and axial directions are zero. The axial stress, σ_x, has been included with the deformation of the cylinder due to water pressure and, therefore, σ_x may properly be neglected when considering the stresses produced by X. It remains to determine how large an error is committed by assuming the radial stress, $\sigma'_r = 0$, since this is true only at the up-stream and down-stream faces. Here, σ'_r is the radial stress produced by X only, since account has already been taken of the radial stress produced by the water pressure and by p_c.

Föppl* gives Airy's stress function† for pure bending of a plane curved beam, Fig. 14, and for the loading indicated in Fig. 15. In deriving the stress function, it is assumed that $\sigma'_x = 0$, that is, that the arch is free to deform vertically as the plane stresses, σ'_t and σ'_r, may require, and that the up-stream and down-stream faces are vertical.

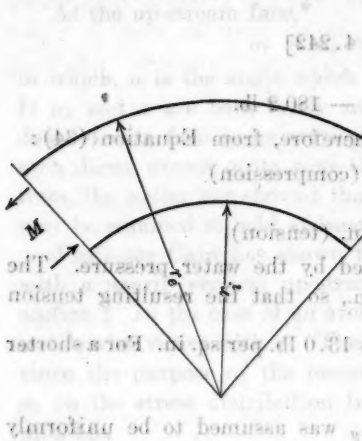

FIG. 14.

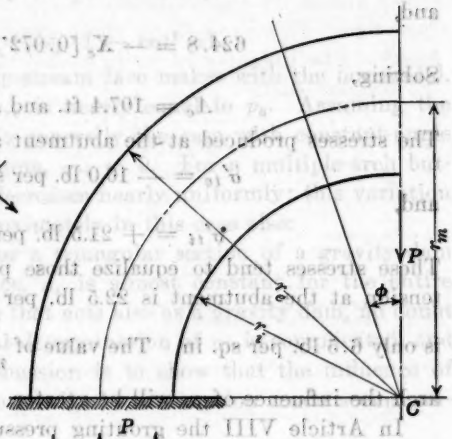

FIG. 15.

Fig. 16 shows the condition here involved, and Fig. 15 may evidently be converted into Fig. 16 by writing $90° - \phi$, in place of ϕ, changing P to $-X$ and, consequently, the moment, $M_o = P\, r_m$, to $M'_o = -X\, r_m$, and adding a moment, $M' = A\, X$. The stresses produced by the loading (Fig. 16), therefore, may be obtained from Föppl's solution by superimposing the conditions of Figs. 14 and 15, that is, by setting $M' = A\, X$, $P = -X$, and $90° - \phi$ for ϕ. The stress function is,

$$F = c_0 + c_1 \log_e r + c_2 r^2 + c_3 r^2 \log_e r + [\alpha r^3 + \beta r \log_e r + \gamma r + \delta r^{-1}] \cos \phi \quad \ldots\ldots(62)$$

in which, r is the distance from the center, C, to any point and r and ϕ are the only variables.

The function, F, must satisfy the fundamental differential equation,

$$\left(\frac{\delta^2}{\delta r^2} + \frac{1}{r^2}\frac{\delta^2}{\delta \phi^2} + \frac{1}{r}\frac{\delta}{\delta r}\right)\left(\frac{\delta^2 F}{\delta r^2} + \frac{1}{r^2}\frac{\delta^2 F}{\delta \phi^2} + \frac{1}{r}\frac{\delta F}{\delta r}\right) = 0 \quad \ldots\ldots(63)$$

The stresses are,

$$\left.\begin{aligned} \sigma_r' &= \frac{1}{r}\frac{\delta F}{\delta r} + \frac{1}{r^2}\frac{\delta^2 F}{\delta \phi^2}; \quad \sigma_t' = \frac{\delta^2 F}{\delta r^2} \\ \tau &= -\frac{\delta}{\delta r}\left(\frac{1}{r}\frac{\delta F}{\delta \phi}\right); \text{ and } \sigma_x' = 0 \end{aligned}\right\} \quad \ldots\ldots(64)$$

* "Drang und Zwang" Bd. I, 2d Edition, p. 302.

† "Transmission of Pressure Through Solids and Soils and the Related Engineering Phenomena," by George Paaswell, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 1563. Mr. Paaswell refers to "A Mathematical Treatise on the Theory of Elasticity," by A. E. H. Love, and to the Appendix of "The Elasticity and Resistance of Materials," by W. H. Burr, M. Am. Soc. C. E.

The axial, that is, vertical, deformation is then,

$$\frac{\delta\,\xi}{\delta\,x} = -\frac{1}{2\,G\,(m+1)}(\sigma_t' + \sigma_r') \quad \ldots\ldots\ldots\ldots\ldots\ldots (65)$$

Since, in general, σ'_t and σ'_r are functions of both r and ϕ, the originally plane horizontal limiting surfaces of the arch do not remain plane under stress. However, Equations (62), (63), and (64) are equally correct if it is assumed that the originally plane horizontal limiting surfaces do remain plane and horizontal; but, in that case, it is required that

$$\sigma_x' = \frac{1}{m}(\sigma_t' + \sigma_r') + \frac{m+1}{m}\,2\,G\,k \quad \ldots\ldots\ldots\ldots\ldots\ldots (66)$$

in which, $k = \dfrac{\delta\,\xi}{\delta\,x}$ is a constant representing the vertical displacement of the horizontal limiting planes. Only when $\sigma'_t + \sigma'_r$ is constant, can σ'_x in Equation (66) remain constant over the entire section.

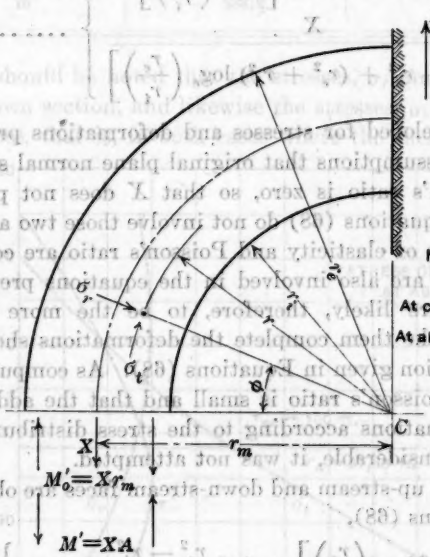

FIG. 16.

XV.—STRESS EQUATIONS

The equations for the stresses are obtained by differentiating Equation (62) in accordance with Equation (64) and determining the constants from the conditions at the surfaces and from the known loading, that is:

For $r = r_e$ and $r = r_i$,

$$\sigma_r' = 0 \text{ and } \tau = 0$$

for $\phi = 0$ (Fig. 16),

$$\left.\begin{aligned} \tau = 0\,;\ \int_{r_i}^{r_e} \sigma'_t\,d\,r = -X\,;\ \int_{r_i}^{r_e} \sigma'_t\,r\,d\,r = -X\,(r_m - A) \end{aligned}\right\} \quad \ldots\ldots (67)$$

and, for $\phi = 90°$,

$$\int_{r_i}^{r_e} \tau\,d\,r = -X\,;\ \text{and}\ \int_{r_i}^{r_e} \sigma_t'\,d\,r = 0$$

This gives for the stresses corresponding to the loading in Fig. 16,

$$\left.\begin{aligned}
\sigma'_r &= \frac{4\,A\,X}{N}\left[-r_i^2\log_e\left(\frac{r}{r_i}\right)-r_e^2\log_e\left(\frac{r_e}{r}\right)\right.\\
&\left.+\frac{r_e^2\,r_i^2}{r^2}\log_e\left(\frac{r_e}{r_i}\right)\right]+2\,Q\cos\phi\left[r-\frac{r_e^2+r_i^2}{r}+\frac{r_e^2\,r_i^2}{r^3}\right]\\
\sigma_t' &= \frac{4\,A\,X}{N}\left[r_e^2-r_i^2-r_i^2\log_e\left(\frac{r}{r_i}\right)-r_e^2\log_e\left(\frac{r_e}{r}\right)\right.\\
&\left.-\frac{r_e^2\,r_i^2}{r^2}\log_e\left(\frac{r_e}{r_i}\right)\right]+2\,Q\cos\phi\left[3r-\frac{r_e^2+r_i^2}{r}-\frac{r_e^2\,r_i^2}{r^3}\right]\\
\tau &= 2\,Q\sin\phi\left[r-\frac{r_e^2+r_i^2}{r}+\frac{r_e^2\,r_i^2}{r^3}\right]
\end{aligned}\right\}\quad\ldots.(68)$$

The constants, N and Q, are given by:

$$\left.\begin{aligned}
N &= (r_e^2-r_i^2)^2-4\,r_e^2\,r_i^2\left[\log_e\left(\frac{r_e}{r_i}\right)\right]^2\\
Q &= -\frac{X}{2\left[r_i^2-r_e^2+(r_e^2+r_i^2)\log_e\left(\frac{r_e}{r_i}\right)\right]}
\end{aligned}\right\}\quad\ldots\ldots(69)$$

All the formulas previously developed for stresses and deformations produced by X were based on the usual assumptions that original plane normal sections remain plane and that Poisson's ratio is zero, so that X does not produce stresses in a radial direction. Equations (68) do not involve those two assumptions, but only that the modulus of elasticity and Poisson's ratio are constant throughout (which assumptions are also involved in the equations previously developed). Equations (68) are likely, therefore, to be the more nearly correct ones, but in order to make them complete the deformations should be computed for the stress distribution given in Equations (68). As computations indicate that the influence of Poisson's ratio is small and that the additional work of determining the deformations according to the stress distribution in Equations (68) may be quite considerable, it was not attempted.

The tangential stresses at the up-stream and down-stream faces are obtained by inserting r_e and r_i in Equations (68),

$$\left.\begin{aligned}
\sigma_{te}' &= \frac{4\,A\,X}{N}\left[r_e^2-r_i^2-2\,r_i^2\log_e\left(\frac{r_e}{r_i}\right)\right]+4\,Q\,\frac{r_e^2-r_i^2}{r_e}\cos\phi\\
\sigma_{ti}' &= -\frac{4\,A\,X}{N}\left[2\,r_e^2\log_e\left(\frac{r_e}{r_i}\right)-(r_e^2-r_i^2)\right]-4\,Q\,\frac{r_e^2-r_i^2}{r_i}\cos\phi
\end{aligned}\right\}\ldots(70)$$

These equations replace Equations (34) and by adding Equations (36) to Equations (70), the resulting stresses are obtained, neglecting temperature effect, etc.

The stress distribution at the abutment of the arch considered in Article XII, computed from Equations (68) and (36) is given in Table 5, and platted in Fig. 17. It will be noted that the tangential stresses at the extrados and intrados agree well with the values found previously (Table 4). Also, the assumption that the shear stress is distributed in accordance with a parabola is at least approximately correct.

The agreement between these stresses found by entirely different methods should give confidence in the correctness of the formulas presented. It has

already been noted that for $\sigma_x = 0$, the writer's formulas agree well with those of Professor Cain. For shorter and thicker arches, the influence of Poisson's ratio, of σ_x, and of p_c, is likely to be more pronounced and for very short arches the Cain formulas are apt to be seriously in error but this latter possibility is likely to be of little practical importance.

TABLE 5.—STRESS DISTRIBUTION AT ABUTMENT.

r	σ_t	σ_r	τ
125	+ 20.5	— 69.5	0.0
120	+ 1.2	— 72.5	17.6
115	— 21.4	— 74.1	34.0
110	— 47.3	— 74.2	48.8
105	— 76.3	— 71.9	61.0
100	— 109.3	— 67.7	69.7
95	— 146.1	— 60.5	74.3
90	— 186.9	— 50.9	72.3
85	— 231.9	— 37.3	62.1
80	— 280.8	— 20.7	39.4
75	— 333.2	0.00	0.0

It should be noted that the stresses, σ_t and σ_r are the principal stresses at the crown section, and likewise the stresses, σ_t, σ_r, and σ_x, at other sections on the faces, that is, in both cases where the shear stress, $\tau = 0$. Regarding the

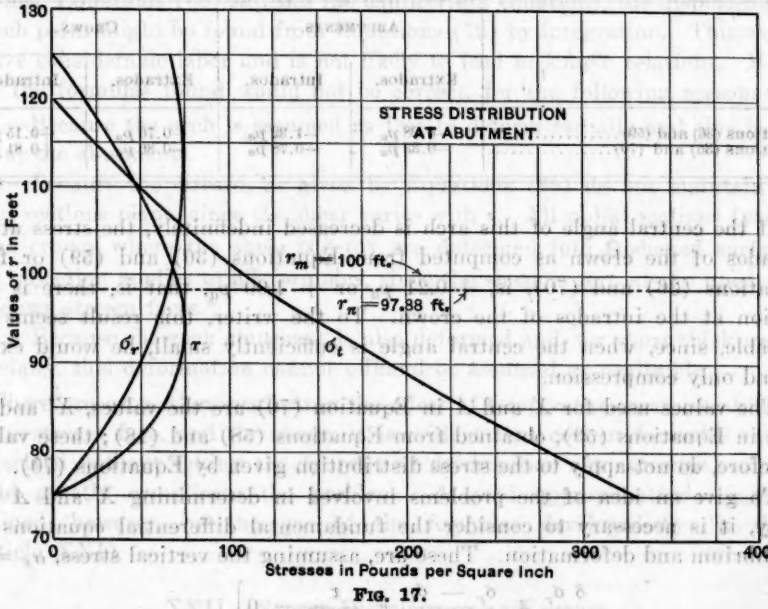

FIG. 17.

stresses which determine failure, the writer discussed these in his paper, "Stresses in Multiple-Arch Dams".* It may be of interest, however, to add that Föppl has subjected various materials, also concrete cubes (1 cm. to 3 cm.) to a hydrostatic oil pressure of about 45 000 lb. per sq. in. and reports† that by

* *Transactions*, Am. Soc. C. E., Vol. LXXXVII (1924), p. 308.

† "Mitteilungen des Münchner Laboratoriums," 27 Heft; see, also, Föppl's "Drang und Zwang," Band I, 2d Edition, p. 41.

far the greater number of specimens showed no evidence whatever of having been submitted to stresses, while a very few had their corners broken. Likewise, several kinds of crystals were subjected to a heavy hydrostatic pressure. They were examined optically before and after the test, as such an examination would reveal even very minor structural changes; but no sign of any change in these crystals was found, although some of them had low compressive strengths when stressed in only one direction. These tests would indicate that for concrete the maximum stress in any one direction is not the deciding factor.

The writer would also call attention to a new theory advanced by Chidô Sunatani in his paper, "Laws of Failure of Solid Bodies Due to Stress".* This theory is an elaboration of Mohr's theory† and seems to the writer to be an improvement.

XVI.—Fundamental Equations

As an example assume $\frac{t}{r_m} = 1.00$ and $2\,\phi_1 = 60°$. Using the Lame stresses in combination with Equations (59) and (70), respectively, the stresses given in Table 6 are found.

TABLE 6.

	Abutments.		Crown.	
	Extrados.	Intrados.	Extrados.	Intrados.
Equations (36) and (59)	$-0.28\,p_a$	$-1.32\,p_a$	$-0.76\,p_a$	$+0.15\,p_a$
Equations (36) and (70)	$-0.33\,p_a$	$-0.78\,p_a$	$-0.86\,p_a$	$+0.81\,p_a$

If the central angle of this arch is decreased indefinitely, the stress at the intrados of the crown as computed from Equations (36) and (59) or from Equations (36) and (70), is, $+0.21\ p_a$ or $+\ 1.00\ p_a$, that is, there is still tension at the intrados of the crown. To the writer, this result seems improbable, since, when the central angle is sufficiently small, he would expect to find only compression.

The values used for X and A in Equation (70) are the values, X' and A', used in Equations (59), obtained from Equations (58) and (18); these values, therefore, do not apply to the stress distribution given by Equations (70).

To give an idea of the problems involved in determining X and A correctly, it is necessary to consider the fundamental differential equations for equilibrium and deformation. These are, assuming the vertical stress, $\sigma_x = 0$:

$$\left.\begin{aligned} \frac{\delta\,\sigma_r}{\delta\,r} + \frac{\sigma_r - \sigma_t}{r} + \frac{\tau}{r\,\delta\,\phi} &= 0 \\ \frac{\delta\,\sigma_t}{r\,\delta\,\phi} + \frac{\delta\,\tau}{\delta\,r} + 2\,\frac{\tau}{r} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (71)$$

* *Technology Reports*, Tôhoku Imperial Univ., Vol. III, No. 1, November, 1922, abstracted by George Paaswell, M. Am. Soc. C. E., in *Engineering News-Record*, Vol. 91, 1923, p. 379.

† "Abhandlungen aus dem Gebiete der Technischen Mechanik" (1906), 5th Abhandlung, p. 187.

When u_r and u_t are the tangential and radial displacements, or change of co-ordinates of any point, (r, ϕ), the equations for the elastic deformations are:

$$\left.\begin{aligned} E\,\frac{\delta u_r}{\delta r} &= \sigma_r - \frac{1}{m}\sigma_t \\ E\left(\frac{\delta u_t}{r\,\delta \phi} + \frac{u_r}{r}\right) &= \sigma_t - \frac{1}{m}\sigma_r \\ G\left(\frac{\delta u_r}{r\,\delta \phi} + \frac{\delta u_t}{\delta r} - \frac{u_t}{r}\right) &= \tau \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (72)$$

Referring to the equilibrium equations, Equations (71), it will be found that the Lamé formulas and Equations (68) satisfy Equations (71); but none of the other formulas does, nor does the formula developed by Professor Cain. In fact, no formula that disregards the radial stress, σ_r, can satisfy the equilibrium equations. This is clear also from the fact that the two tangential stresses which act on an element, $r\,d\phi\,d_r$, are not parallel and therefore there must be a component in the radial direction. At the crown the shear is zero and hence σ_r cannot be zero. For slender arches the error introduced by neglecting σ_r is unquestionably insignificant, while for short thick arches this error may be considerable.

Since Equations (68) satisfies the equilibrium equations, the displacements of each point might be found from Equations (72) by integration. This would involve considerable labor and is not likely to lead to simple relations. Moreover, the formulas found would not be correct, for the following reasons:

1.—Because the arch is assumed as free to deform radially and this is not true at the abutments.

2.—Because the stresses as given by Equations (68) do not maintain the radial sections plane, since the shear varies with r. All radial sections (except at the crown, where the shear is zero), are deformed into S-shaped surfaces. This objection applies to all formulas, since the shear is zero at the up-stream and down-stream faces.*

3.—Because the rock abutment is also deformed and, for short thick arches especially, this deformation cannot offhand be assumed as negligible.

Therefore even when neglecting the effect of temperature, water-soaking of the up-stream face, and shrinkage, it is evident that correct formulas cannot be derived without taking the foundation deformation into account and this would greatly complicate the problem. A good approximation might be obtained, however, by the use of the formulas developed by Dr. Vogt (Article XXI).

XVII.—Stress Distribution at Failure.

Lamé's formulas and the formulas developed in Article XV for the stresses caused by X assume that the modulus of elasticity, E, and Poisson's ratio, m, are constant. This is approximately true for stresses well beyond the ordi-

* In the paper by Dr. Vogt, referred to in Article XXI, it is shown that a moment warps the surface of a rock foundation into an S-shaped curve.

nary allowable working stresses, up to, perhaps, one-third the ultimate compressive strength.*

A considerable variation of Poisson's ratio will not change the results materially, but a variation in E, such as takes place as the stresses approach their ultimate values, will influence the stress distribution materially.

Tests made on cast iron by Professor von Bach† throw some light on this subject, since for cast iron E also varies considerably with the stress. He experimented on machined cast iron and found its tensile strength to be 1 315 kg. per sq. cm. When testing this cast iron in bending, by supporting it at the ends and loading it in the center, Bach found that the bending moment at failure was 1.64 times greater than the value computed from the ordinary formula:

$$s_t = \frac{M}{I}\,\frac{t}{2}$$

in which, s_t is the tensile strength and t the thickness of the beam. The stress-strain curve for this cast iron is shown in Fig. 18. From the fact that the total compression must equal the total tension and from the known thickness of the test specimen, the neutral axis can be located by means of a planimeter. The centers of gravity of the two areas are then determined. If the distance between them is Y and the total compression area represents N kg., the moment at failure would be $N\,Y$, on the assumption that normal planes remain plane. Bach found the moment thus computed to agree with bending tests to within 3 per cent. The error in assuming E constant in this case results in underestimating the factor of safety in the ratio of 1.64 to 1.

A stress-strain curve for concrete in compression is shown in Fig. 20.‡ This shows that the deformations increase faster than the stresses and the more so the greener the concrete. Other experimenters find the same to be true, but the actual shape of the curve seems to depend on the age, ultimate strength, mix, water-cement ratio, etc. The variation of E will affect the stress distribution found by Lamé's formulas, decreasing the difference between the stresses at the extrados and intrados.§ It will also affect the distribution of the stresses due to X. A picture of what takes place may be formed by assuming that in Equation (29) for y', E associated with I becomes smaller, while E associated with t remains constant. Also, both k_e and k_i are decreased and the stress distribution in bending is as indicated in Fig. 19.

In order to obtain an idea of how these modifications will influence the stresses, assume arbitrarily, in the example given in Article XII, that the Lamé stresses from Equations (36) are increased 20 lb. per sq. in. for the extrados and decreased 20 lb. per sq. in. for the intrados. Assume, also, that E associated with I is reduced to one-third its former value, while E associated

* See, for example, Hool and Johnson, "Concrete Engineer's Handbook," 1st Edition (1918), Fig. 26, p. 250; also, "Modulus of Elasticity of Concrete," by Stanton Walker, *Bulletin 5*, Structural Materials Research Laboratory, Lewis Inst., Chicago, Ill., Fig. 3, p. 16.

† "Elasticität und Festigkeit," 9th Edition, p. 298.

‡ Taylor and Thompson, "Concrete, Plain and Reinforced," 2d Edition (1909), Fig. 129, p. 403.

§ Bach, "Elasticität und Festigkeit," 9th Edition, p. 579.

with t remains unaltered. Then, y' increases to $\frac{3.706}{E}$. Since ϵ' and ϵ'' are nearly alike, it may be assumed that ϵ' remains unchanged; X' is therefore decreased in proportion as y' is increased, or the new value of X' is 70.5% of its former value. Further, let it be arbitrarily assumed that both k_e and k_t are reduced to 85% of their former values. Then when $\sigma_x = -300.0$ lb. per sq. in., the stresses at the crown become:

$$\sigma_{te} = -167.7 + 1.5 = -166.2 \text{ lb. per sq. in. (compression)}$$

$$\sigma_{ti} = -197.2 + 118.2 = -79.0 \text{ lb. per sq. in.}$$

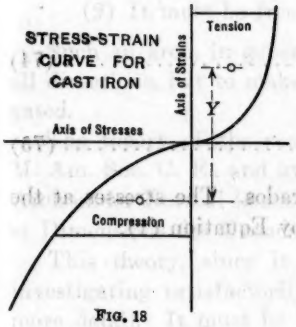

FIG. 18

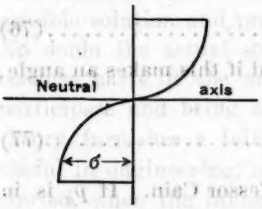

FIG. 19.

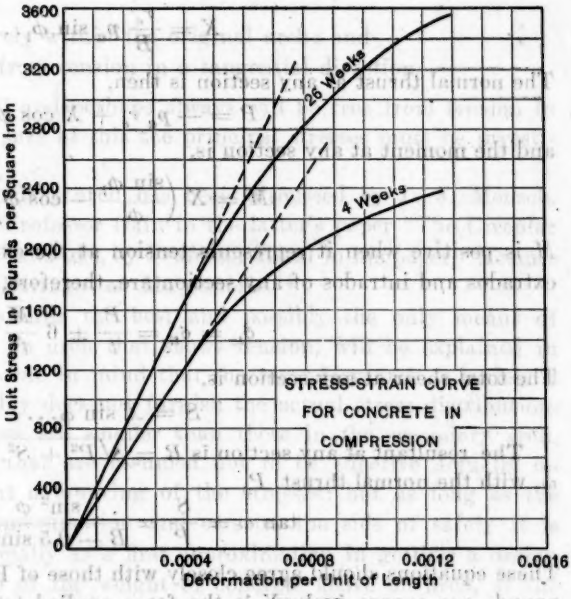

FIG. 20.

and at the abutments,

$$\sigma_{te} = -167.7 + 107.2 = -60.5 \text{ lb. per sq. in.}$$

$$\sigma_{ti} = -197.2 - 68.5 = -265.7 \text{ lb. per sq. in.}$$

If the stress were uniformly distributed, it would be:

$$\sigma \text{ (cylinder)} = -\frac{69.5 \times 125}{50} = -174.0 \text{ lb. per sq. in.}$$

Comparing the stresses here obtained with those given in Table 4, it will be seen that the maximum compression has been lowered from 340.1 to 265.7 lb. per sq. in.; but this is still about 50% higher than the stress computed from the cylinder formula. The tension which existed at the extrados of the abutments, as shown in Table 4, has been converted into compression. It seems obvious that the effect of having E decrease with the stress is to bring about a more uniform stress distribution than with E constant. A thick arch tested to destruction therefore should show a considerably higher factor of safety than

is indicated by calculations based on E as a constant. The assumption that the cylinder formula holds good, however, errs on the side of danger; no conservative engineer can afford to make it.

XVIII.—Approximate Formulas

If,

$$\epsilon'' = -\frac{p_a r_e}{E t}$$

from Equation (38 b) is substituted for ϵ' from Equation (38), that is, if σ_x is assumed to be zero, there results from Equation (58):

$$X = \frac{r_e}{B} p_a \sin \phi_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (73)$$

The normal thrust at any section is then,

$$P = -p_a r_e + X \cos \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (74)$$

and the moment at any section is,

$$M = X \left(\frac{\sin \phi_1}{\phi_1} - \cos \phi \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (75)$$

M is positive when it represents tension at the extrados. The stresses at the extrados and intrados of any section are, therefore, by Equation (1),

$$\sigma_{te} = \sigma_{ti} = \frac{P}{t} \pm 6 \frac{M}{t^2}$$

The total shear at any section is,

$$S = X \sin \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (76)$$

The resultant at any section is $R = \sqrt{P^2 + S^2}$, and if this makes an angle, ω, with the normal thrust, P,

$$\tan \omega = \frac{S}{P} = \frac{\sin^2 \phi}{B - 0.5 \sin^2 \phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (77)$$

These equations should agree closely with those of Professor Cain. If p_a is in pounds per square inch, X is the force applied to an arch slice $\frac{1}{12}$ in. thick.

The angle, ω, between the total thrust and the total resultant is a fictitious angle. The real angle between the resultant stress and the normal stress at any point varies from point to point of a radial section; it is zero at the up-stream and down-stream faces, since the shear is zero, and it reaches a maximum somewhere near the middle point of the section. At the abutment of the arch investigated in Article XV (see Table 5), for $r = 95$, $\tau = 74.3$ and $\sigma_t = -145.1$, so that $\tan \omega = 0.511$ and $\omega = 27$ degrees.

From Equation (45)) (substituting ϵ'' for ϵ') and Equation (60) the crown deflection is,

$$\eta_t = \frac{r_e r_m}{E_t} p_a \left(\frac{D}{B} \sin \phi_1 + 1 - \cos \phi_1 \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (78)$$

Professor Cain's equation in connection with his Table 4* and Fig. 9† is more convenient. The results from both methods should agree closely.

* *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 264.
† *Loc. cit.*, p. 265.

XIX.—Secondary Arch

It frequently happens in the design of arch dams that some of the arches cannot be given a sufficiently large central angle for a proper ratio of $\frac{t}{r_m}$ and that, therefore, tension is found at the abutment, or perhaps both at the abutment and at the crown. The question then necessarily arises as to what the maximum compression in such an arch is, assuming that the concrete does not transmit any tension.

In this case a secondary arch may be conceived, which fulfills the following conditions:

(1) It must lie entirely within the original arch; and

(2) It must be free from tension in a tangential direction.

Such an arch, in general and perhaps always will be free from tension in all directions, but to make sure of this the principal stresses must be investigated.

The theory of the secondary arch has been discussed by L. J. Mensch, M. Am. Soc. C. E., and by Professor Cain in the latter's paper "The Circular Arch under Normal Loads"* and also by Professor Résal in his paper, "Formes et Dimensions des Grands Barrages en Maçonnerie,"†

This theory, since it affords the best and possibly the only means of investigating satisfactorily an arch that shows tension, will be explained in more detail. It must be borne in mind that the secondary arch is merely a possible solution and probably does not furnish the actual stress distribution. No doubt the actual stresses are smaller than those in the secondary arch, because parts of the arch that are assumed not to be effective actually do participate and bring about a lowering of the stresses; but as long as the theory furnishes a fair approximation and errs on the side of safety it is useful in engineering, especially as a first approximation in getting a design started, when the influence of the weight of concrete above an arch, of the temperature, swelling, shrinkage, and cantilever action must be neglected in order to simplify the problem.

On the assumption that the vertical downward pressure, σ_x, is zero, the stress at the extrados of the abutment is, from Equations (36) and (59):

$$\sigma_{te(ab)} = -\left(\frac{r_m}{t} + \frac{1}{4}\frac{t}{r_m}\right) p_a$$

$$-\frac{\epsilon'}{B} E \sin \phi_1 \left\{ k_e \frac{r_m}{t}\left[\frac{\sin \phi_1}{\phi_1}\frac{r_n}{r_m} - \cos \phi_1\right] + \cos \phi_1 \right\} \quad \ldots\ldots (79)$$

The value of ϵ' is obtained from Equation (38) by putting $\sigma_x = 0$. If $\sigma_{te(ab)} = 0$ in Equation (79), there may be found for any value of $\frac{t}{r_m}$ a corresponding limiting value, $2\phi_0$ for the central angle, $2\phi_1$. For any

* *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 249 and p. 272, foot note.

† *Annales des Ponts et Chaussées*, II, 1919.

larger value of the central angle, there is compression at the extrados of the abutment, and for a smaller value, tension at the abutment and perhaps also at the crown. Let the limiting ratio for $\frac{t}{r_m}$ be called x_0, then the corresponding values, as determined from Equation (79), are given in Table 7 and plotted in Fig. 21.

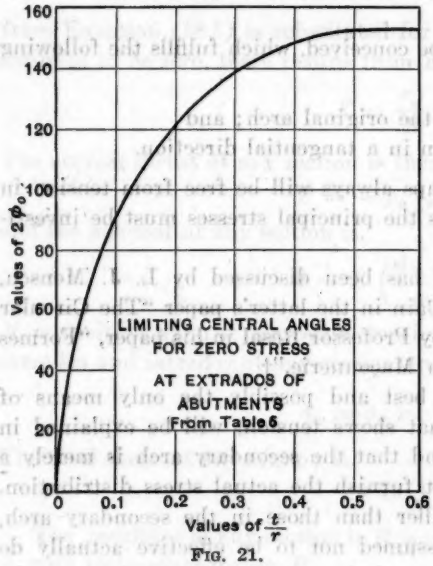

FIG. 21.

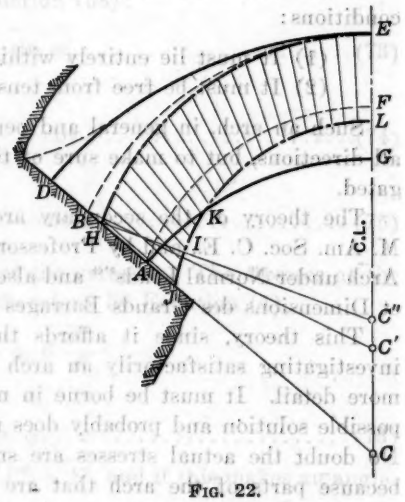

FIG. 22.

TABLE 7.—CORRESPONDING LIMITING VALUES OF $x_0 = \frac{t}{r_m}$ TIMES CENTRAL ANGLE.

x_0 =	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.50	0.60
$2\phi\ 4_0$ =	68.5°	92.5°	106°	121.5°	130°	138°	144°	148°	151°	152°

To illustrate, assume an arch with $\frac{t}{r_m} = 0.353$ and $2\ \phi_1 = 100°$, as shown to scale in Fig. 22. As these values locate a point in Fig. 21 which lies below the curve, this arch will develop tension at the abutments when $\sigma_x = 0$. By drawing a few secondary arches, the secondary arch which has no stress at the extrados of the abutment can be quickly located by referring to Fig. 21. In the case under consideration, the corresponding values are $x_0 = \frac{16.5}{65.5} = 0.252$ and $2\,\phi_0 = 132$ degrees. The stress in this arch is zero at Point B; at Point

A, where the maximum compression exists, it is nearly (as will be shown later):

$$\sigma_{max.} = -2\frac{r_e^2}{t}\frac{p_a}{r_m} = -\left(1+\frac{2}{x_0}+\frac{x_0}{2}\right)p_a$$

or,

$$x_0 = \frac{5}{2\dfrac{\sigma_{max.}}{p_a} - 1} \qquad \text{.....(80)*}$$

The maximum stress is nearly equal to $2\frac{r_e}{r_m}$ times the stress found by the cylinder formula, when this is applied to the secondary arch. In this case it is,

$$\sigma_{max.} = -10.95\,p_a\text{†}$$

It may happen that this value is considered excessive by the designer. In the case of a new design, he can lower this maximum stress by adding to the intrados at the abutment, as indicated in Fig. 22; it is assumed, of course, that r_e cannot be decreased. This change increases x_0 and calls for a corresponding increase of the limiting central angle. By moving the center to C'', the new value for the central angle is 144° and from Fig. 21 the corresponding value of x_0 is 0.35 and the maximum compression becomes only $-7.56\,p_a$ instead of $-10.05\,p_a$. The up-stream radius of the secondary arch, scaled from the drawing, is 67 ft. and this gives $r_m = 57$ ft. and $t = 20.0$ ft.

In this case the arch carrying the load is $E\,H\,J\,K\,L$ and p_a is assumed to act on $E\,H$. If the original arch actually cracks at $D\,H$ permitting water pressure to act on $D\,H$, the resultant of p_a acting on $E\,H$ is equal to the resultant of p_a acting on $E\,D$ and $D\,H$. Due to the swelling of the up-stream face produced by water-soaking, the writer believes that the up-stream face actually will not crack and that the stresses found from Equation (80) are in excess of the actual stresses.

In case the swelling is not sufficient to relieve all tension, the secondary arch actually formed will be thicker at the abutment than at the crown, at least for thin and medium thick arches. For example, it might be such as $E\,B\,J\,K\,L$ in Fig. 22, the stress at Point B being zero. The stresses in such an arch may be investigated as will be shown in Article XX, but this involves much more labor and would not result in lowering them materially.

It should be noted that it is unnecessary to continue the arch, $L\,K\,I$, in Fig. 22 beyond the point, I, and it is sufficient to draw $I\,J$ perpendicular on the radius, $H\,I\,C''$. This is so because with the stress at Point H equal to zero, the angle that the resultant makes with the perpendicular on the radius, $H\,I\,C''$, is always small and the resultant acts one-third of the distance, $I\,H$, from I; if this resultant intersects the abutment between H and J, nothing is gained by continuing the curve beyond Point I. From Equation (77) the

* r_e, r_m, t, and x_0 apply to the secondary arch.

† It is interesting to note that the stress computed by the cylinder formula applied to the original arch gives $-3.38\,p_a$, the error amounting to 300% on the side of danger. The cylinder formula, therefore, can hardly be considered as a useful approximation. This case is by no means extreme.

angle, ω, for values of $\frac{t}{r_m}$ of 0.1, 0.3, and 0.6 will be 3°, 6°, 30′, and 12° 40′, respectively.

The maximum compression in a secondary arch occurs at the abutments and may be determined as follows, if ϵ'' is substituted for ϵ':

$$\left.\begin{aligned}\sigma_{t\,e\,(a\,b)} &= -\left(\frac{1}{x_0}+\frac{1}{4}x_0\right)p_a+\frac{p_a r_e}{t B}\sin\phi_1 \\ &\quad \left\{\frac{k_e}{x_0}\left[\frac{\sin\phi_1}{\phi_1}\frac{r_n}{r_m}-\cos\phi_1\right]+\cos\phi_1\right. \\ \sigma_{t\,i\,(a\,b)} &= -\left(1+\frac{1}{x_0}+\frac{x_0}{4}\right)p_a-\frac{p_a r_e}{t B}\sin\phi_1 \\ &\quad \left\{\frac{k_i}{x_0}\left[\frac{\sin\phi_1}{\phi_1}\frac{r_n}{r_m}-\cos\phi_1\right]-\cos\phi_1\right.\end{aligned}\right\} \quad \ldots\ldots (81)$$

For a secondary arch, subtracting $\sigma_{t\,e\,(a\,b)}$ (equal to zero) from $\sigma_{t\,i\,(a\,b)}$, since $k_e + k_i$ is nearly equal to 12, and assuming that $\frac{r_n}{r_m} = 1$,

$$\sigma_{t\,i\,(max.)} = -p_a - \frac{12}{B}\frac{r_e r_m}{t^2}\left[\frac{\sin^2\phi_1}{\phi_1}-\frac{1}{2}\sin 2\phi_1\right] \quad \ldots\ldots (82)$$

The values for $\sigma_{max.}$ and $\sigma_{t\,i\,max.}$ are given in Table 8; it will be seen that Equation (80) is a good approximation for values of $\frac{t}{r_m}$ up to 0.6. The deflection of the secondary arch may be found from Equation (78).

TABLE 8.—Comparison of Stresses from Equations (80) and (82).

$\frac{t}{r_m}$ =	0.1	0.2	0.3	0.4	0.5	0.6
$\sigma_{max.}$ =	$-22.3p_a$	$-12.1p_a$	$-8.81p_a$	$-7.2p_a$	$-6.25p_a$	$-5.63p_a$
$\sigma_{t\,i\,(max.)}$ =	$-20.6p_a$	$-11.1p_a$	$-8.91p_a$	$-7.13p_a$	$-6.34p_a$	$-5.89p_a$

The formulas given by Professor Résal are based on a parabolic arch. They check Equations (80) and (82) for secondary arches fairly well, are more convenient to use, and will, therefore, be given here. Referring to Fig. 23, in which the hatched portion is the parabolic secondary arch, the maximum compression, which occurs at the intrados of the abutment, is,

$$\sigma_{max.} = -1.41\,p_a\frac{L^2}{F^2} \quad \ldots\ldots (83)$$

The crown deflection is,

$$E\,\eta_a = 0.209\,\frac{L^4}{F^3}\,p_a \quad \ldots\ldots (84)$$

The radial thickness, t', of the secondary arch, and the values of F and L (Fig. 23) are:

$$\left.\begin{aligned} t' &= \frac{F}{4.75} \\ F &= t + r_i\cos\phi_i \\ L &= 2\,r_e\sin\phi_e\end{aligned}\right\} \quad \ldots\ldots (85)$$

It is generally better to draw the arch to scale and then measure L and F; care should be taken to draw the abutments on both sides as they actually occur for each arch to be investigated.

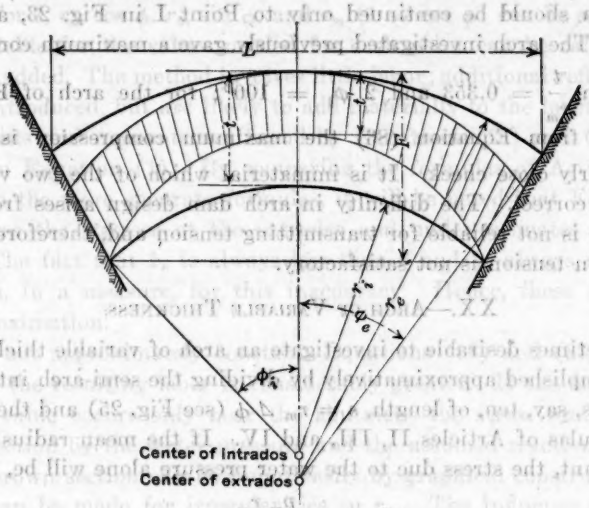

FIG. 23.

If the maximum stress is found to be higher than is thought advisable and if r_e cannot be made smaller without producing excessive overhang in the dam, F must be made larger. It can be determined from Equation (83), if the maximum stress is assumed. The corresponding thickness is $t' = \frac{F}{4.75}$ and a parabola must be constructed in order to find the shape of the down-stream face, although, in general, a circle is a sufficiently close approximation.

Referring to Fig. 24, the equation of a parabola is:

$$y^2 = 2\,p\,x$$

The value found for F locates a point on the abutment a distance, H, from the center line, where,

$$H = (t + r_i - F) \tan \phi_i$$

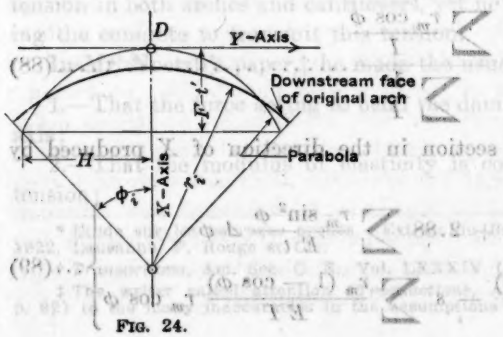

FIG. 24.

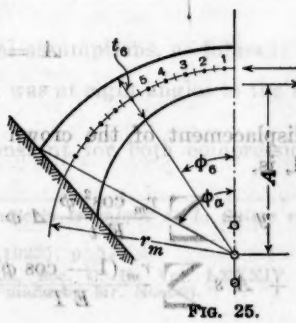

FIG. 25.

For this point, $y = H$ and $x = F - t'$; therefore,

$$2p = \frac{H^2}{F - t'}$$

The parabola should be continued only to Point I in Fig. 23, as explained previously. The arch investigated previously gave a maximum compression of $10.05\ p_a$ with $\frac{t}{r_m} = 0.353$ and $2\ \phi_1 = 100°$; for the arch of Fig. 24, $\frac{L}{F} = 2.79$ and from Equation (83) the maximum compression is $10.95\ p_a$. This is a fairly close check. It is immaterial which of the two values is the more nearly correct. The difficulty in arch dam design arises from the fact that concrete is not reliable for transmitting tension and, therefore, a solution that results in tension is not satisfactory.

XX.—Arch of Variable Thickness

It is sometimes desirable to investigate an arch of variable thickness. This may be accomplished approximatively by dividing the semi-arch into a number of equal parts, say, ten, of length, $s = r_m \Delta \phi$ (see Fig. 25) and then by applying the formulas of Articles II, III, and IV. If the mean radius is approximately constant, the stress due to the water pressure alone will be,

$$\sigma'' = -\frac{p_m r_m}{t} \qquad (86)$$

in which, p_m is the water pressure, referred to the center line. The thrust, $p_m r_m$, acting at the middle points of the crown section and of the abutment section, holds the water pressure in equilibrium. These two reactions and the water pressure produce a displacement, DF, in Fig. 2 and the force, X, pulls the crown section back into the center line. Let the displacement, $DF = D$, then,

$$D = \sum \frac{p_m r_m}{E t} \cos \phi \qquad (87)$$

In order to find A (by Equation (4)) since the moment of inertia, I, is not now a constant,

$$\sum \frac{M}{I} s = 0$$

M is given by Equation (5), so that,

$$A = \frac{\sum \frac{r_m \cos \phi}{I}}{\sum \frac{1}{I}} \qquad (88)$$

The displacement of the crown section in the direction of X produced by $X = 1$, is,

$$\left.\begin{aligned} y = {} & \sum \frac{r_m \cos^2 \phi}{E t} \Delta \phi + 2.88 \sum \frac{r_m \sin^2 \phi}{E t} \Delta \phi \\ & + A s \sum \frac{r_m (1 - \cos \phi)}{E I} - s \sum \frac{r_m (1 - \cos \phi)}{E I} r_m \cos \phi \end{aligned}\right\} \qquad (89)$$

and,

$$X = \frac{D}{y} \quad \ldots\ldots\ldots\

3.—That the modulus of elasticity is the same for compression as for tension;

4.—That the deformation due to shear is negligible;

5.—That the influence of lateral deformation (Poisson's ratio) is negligible;

6.—That the foundation is unyielding; and

7.—That the water-soaking of the up-stream face does not increase the volume of the concrete, that is, does not cause swelling.

The writer discussed these assumptions in detail and showed that they were so approximate for dams that they could not furnish a sound basis for the sweeping conclusions made by Mr. Noetzli, more especially since these conclusions were not supported by any facts whatever, but on the contrary appeared to be at variance with facts.

It seems to the writer that the cantilever cannot relieve its tension by a single crack as Mr. Noetzli suggests,* but that it must yield a little over a considerable vertical zone, so as to afford the arches an opportunity to carry the load, which actually they were doing if Mr. Noetzli's thesis was correct, as the dams had not failed nor shown any tendency in that direction. It is probable that a secondary cantilever would be formed in much the same manner as the secondary arches. This secondary cantilever can probably be determined by trial computations, but the labor might be considerable. At any rate a solution such as that offered by Mr. Noetzli, which showed tension much in excess of what could exist, is not satisfactory, since it is an impossible condition.

That the effect of swelling of the up-stream face (according to Mr. Noetzli there is no swelling) is not by any means negligible is illustrated by the Santa Anita Dam, a constant angle arch dam now being constructed by the Los Angeles County Flood Control District. This dam is 225 ft. high and of dimensions as given in Table 9. Assume the dam to have been closed or pressure grouted after the concrete was well dried out; the shrinkage would probably amount to 0.0002. Water against the up-stream face would cause a swelling such that the volume would be 0.0001 larger than when poured, that is, a relative swelling of 0.0003. It seems fair to assume that a relative swelling of at least 0.0001 occurs; this is no doubt an under-estimate, since for a fairly thick dam the down-stream face probably does not participate in the swelling to any marked degree. A swelling of 0.0001 is equivalent to imposing a tension of 200 lb. per sq. in. on the up-stream face, on the assumption that $E = 2\,000\,000$ lb. per sq. in. The deflection caused by this swelling is therefore considerably greater than the cantilever could otherwise make without failing in tension. The swelling of the up-stream face, besides producing a down-stream deflection of the cantilever, also produces an up-stream deflection of the arches. This latter is not readily computed because the arches in the Santa Anita Dam are thicker at the abutments than at the crown (except near the top), and it might also be necessary to determine new secondary arches, which would be thicker than the secondary arches carrying the load when there is no swelling. The following computation, however, may furnish

* *Transactions*, Am. Soc. C. E., Vol. LXXXIV (1921), p. 54.

an indication of the influence of swelling. Consider the arch at elevation 1280; from Equation (38 *b*), if $E = 2\,000\,000$,

$$\epsilon'' = -\frac{19.5 \quad 287.0}{E \; 18} = -\frac{1.55}{10\,000}$$

If a swelling equal to ϵ'' occurs uniformly throughout the arch (see Equations (53) and (55)), the arch would not deflect at all when the water load was imposed. At Elevation 1220, using an average value of 40 ft. for t in Equation (38 *b*), $\epsilon'' = -0.000163$. If the swelling amounts to as much as this, the arch would not deflect at all when taking its full load.

TABLE 9.—DATA FOR SANTA ANITA DAM.

Elevation.	t_c, in feet.	t_a, in feet.	$2\,\phi_1$*, in degrees.	r_a, in feet.	v, in feet.	η_a, in feet.
1325	7.0	7.0	115	305	0.075	0.0000
1300	12.5	12.5	106	292	0.056	0.0287
1280	18.0	18.0	101	287	0.043	0.0503
1260	23.0	27.0	94	274	0.033	0.0695
1240	26.0	47.0	90	269	0.024	0.0816
1220	29.0	50.0	79	254	0.017	0.1015
1200	32.0	60.0	65	238	0.011	0.0886
1180	34.0	63.0	55	214	0.0066	0.0608
1160	38.5	60.0	47	190	0.0028	0.0292
1140	42.0	48.0	32	158	0.0010	0.00466
1120	45.0	46.5	30	135	0.0004	0.00199
1100	48.0	48.0	30	109	.....	.....

* Measured from the center for the up-stream radius to the junction of the extrados with the abutments.

The cantilever deflections were computed as follows, referring to Fig. 26, in which, θ denotes the rotation brought about at the points, 1, 2, 3, . . . n, by the swelling, t denotes the average thickness, s, the lengths of each section, and v, the down-stream deflections at the points, 1, 2, 3, . . . n. For a swelling of 0.0001,

$$\theta_1 = \frac{0.0001\; s_1}{t_1}$$

$$\theta_2 = \theta_1 + \frac{0.0001\; s_2}{t_2}$$

$$\theta_n = \theta_{n-1} + \frac{0.0001\; s_n}{t_n}$$

and the cantilever deflections,

$$v_1 = 0.5\; s_1\; \theta_1$$

$$v_2 = v_1 + s_2\; \theta_1 + 0.5\; s_2\; (\theta_2 - \theta_1)$$

$$v_n = v_{n-1} + s_n\; \theta_{n-1} + 0.5\; s_n\; (\theta_n - \theta_{n-1})$$

These deflections are given in Table 9, also the deflections of the secondary arches, on the assumption that they carry the entire water load. The arch deflections were computed by Résal's method, Equation (84). The two deflection curves are shown to scale in Fig. 27, from which it is quite obvious that the deflection of the cantilever caused by the swelling is by no means negligible. It is considerably greater than the cantilever deflection obtained by computing

only for bending as was done by Mr. Noetzli. The influence of this cantilever deflection due to swelling is likely to be comparatively greater at other parts of the dam.

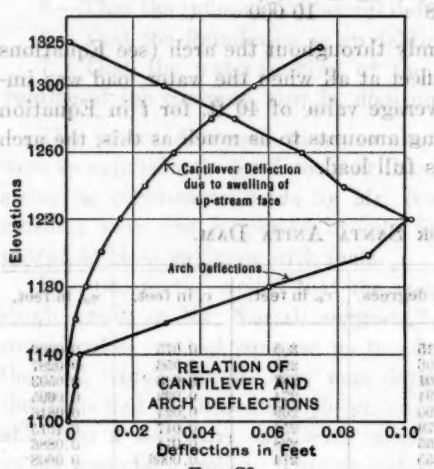

FIG. 26.

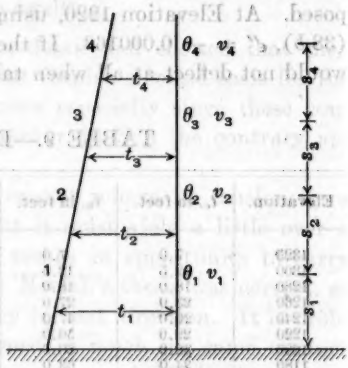

FIG. 27.

The writer* further disagreed with Mr. Noetzli and asserted that the deformation of the foundation might contribute materially to the deflection of the cantilever. In a thesis kindly furnished by a Norwegian engineer, Dr. Frederik Vogt, the deformations of rock foundations are investigated.† Dr. Vogt considers a gravity dam, having a base two-thirds of its height and a length five times the height. The usual triangular section is evidently considered, although the author does not say so specifically, but his figures for deflection of the cantilever checks the writer's calculations.‡ Poisson's ratio is assumed as 0.167. Dr. Vogt gives the deflection in feet at the crest of the cantilever, when sustaining the full water pressure, as

$$f_1 = 4.5\, w\, \frac{h^2}{E}$$

in which, $w = 62.5$, the weight of a cubic foot of water, h, the height of the dam, in feet, and E is in pounds per square foot. About three-fourths of this is due to moment and one-fourth to shear.

The crest deflection, in feet, caused by the deformation of the foundation alone, is, according to Dr. Vogt,

$$f_2 = 3.74\, w\, \frac{h^2}{E}$$

Thus, the crown deflection produced by the yielding of the foundation alone amounts to about 85% of the crest deflection due to the deformation of the cantilever alone.§ Such a percentage can hardly be called negligible.

* *Transactions*, Am. Soc. C. E., Vol. LXXXIV (1921), p. 91.

† "Ueber die Berechnung der Fundament-deformation," Det Norske Videnskaps Akademi, Oslo, Norway, Math.-Naturv. Klasse 1925, No. 2.

‡ *Transactions*, Am. Soc. C. E., Vol. LXXXIV (1921), Fig. 32, p. 101.

§ Dr. Vogt assumes E to be nearly the same for rock and concrete.

Dr. Vogt investigates the influence of a normal force, a moment, and a shear separately, and has derived simple approximate formulas. The paper merits a study by any one seriously interested in dams or other massive structures.

The yielding of the foundation will also increase the deflection of the arches, but the writer believes to a much smaller extent, because the arches are fixed at the crown and not free to follow the deformations at the abutment.

It will be evident that the assumptions usually made, in finding the division of load between the cantilevers and the arches, are not sufficiently accurate to warrant any conclusions. The writer would like to see some fairly reliable formulas developed, which would take account of all the factors of the same order of importance. Until this can be done he prefers to neglect the influence of the cantilevers and make sure that the arches can support the full water pressure with a large margin of safety. He does not deny the influence of the cantilevers, but it is obviously without avail to select arbitrarily one of the many factors, and not the largest one at that, and neglect the influence of all the others.

That the standard assumptions are correct for standard conditions of ordinary structures is not denied, and moreover that question is not here involved; the question is whether they are acceptable for those conditions involved in designing dams.

It is not clear how neglecting the cantilever action can mean that the actual factor of safety of the dam is less than the factor calculated on the assumption that the cantilevers carry no part of the load. There may be calculated tensile stresses, but, as long as the arches can carry the full water load with a great margin, it is not evident how such a dam can possibly fail or become a dangerous structure.

The voice of experience is also entitled to be heard, and it indicates that even poorly designed and constructed single arch dams are relatively safe structures.* Theory supports this view. The fact that no single arch dam has ever failed, although several freak single arch dams exist, is also to be given some weight, especially in view of the fact that this type is the only one that has not produced a fair number of failures.†

Acknowledgment

This paper was submitted for criticism to Professor William Cain, to whom the writer's sincere thanks are due for discussing it and encouraging its submission to the profession.

* See A. J. Wiley, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. LXXXIII (1919–20) p. 609. The radius at the top varies between 58 ft. and 80 ft. In cold weather, with reservoir empty, a crack of ⅜ in. appeared; this closed when the reservoir filled.

† "The Record of 100 Dam Failures," L. R. Jorgensen, M. Am. Soc. C. E., *Journal of Electricity*, March 15 and April 1, 1920.

STRESSES IN THICK ARCHES OF DAMS

Discussion*

By William Cain, M. Am. Soc. C. E.

William Cain,† M. Am. Soc. C. E. (by letter).‡—The author has effected a solution of the thick arch by a geometrical method that is of decided value. For Poisson's ratio, $\frac{1}{m} = 0$, the results agree with those derived by the "work method", as given subsequently. When σ_x, the vertical unit stress, is considered, and m is finite, the author's method leads to a solution which is nearly as simple as the preceding and is not included in the writer's formulas.

After the publication of his paper on "The Circular Arch Under Normal Loads,"§ the writer began investigating the subject of thick arches and sent, in 1922, certain results to interested friends. Finally, when Mr. Jakobsen kindly sent the first draft of his paper in manuscript form, the writer's interest was aroused and he completed a thorough treatment of the thick arch by the work method as herein given. To make it convincing, every preliminary step was proved from elementary principles; then the work method was applied, giving X' and the deflection, which, moreover, agree with the author's values, for $m = \infty$. In this investigation, the most signal advance was made by referring moments and thrusts to the neutral axis. The writer was first led to this by the demonstration given later in connection with Fig. 33, although another briefer proof is first given. Using as a base the neutral axis, the general formula for L, the elastic work of the half-arch was expressed in as simple a form as in the writer's paper cited, so that the subsequent derivations were almost as simple as before. The analysis follows.

Tangential Stresses.—Let Fig. 28 represent a part of the arch 1 ft. thick perpendicular to the plane of the paper, bounded by two right sections making, with each other, the angle, α, with the center line of radius, r, passing through C and the neutral line of radius, r_n, passing through D. Let, $r - r_n =$ distance $O\ C -$ distance $O\ D =$ distance $D\ C = c$. Since the full water pressure is carried partly by the arch and partly by the corresponding cantilever, let p_e equal the unit water pressure carried by the arch and applied at its extrados. Suppose that the equilibrium polygon for the full arch fixed at the ends and subjected to the forces, p_e, passes through Point E, where the components (normal and along the section) of the resultant are P and S, re-

* This discussion (of the paper by B. F. Jakobsen, M. Am. Soc. C. E., published in this number of *Proceedings*) is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

† Prof. Emeritus, Univ. of North Carolina, Chapel Hill, N. C.

‡ Received by the Secretary, December 2, 1925.

§ *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 233.

spectively, the latter being the shear. Then, on applying at D two forces, P, equal and opposed and both normal to the section, the single force, P, at E is now replaced by the couple of moment, P times the distance, $D\,E$, the single force, P, at D on the neutral axis tending to compress the section. It will be shown later that this force, P, at D causes only a translation of the section to a parallel position and that the rotation of this section is caused entirely by the couple of moment, $M = P$ times the distance, $D\,E$. This would not be true if the two opposed forces P, were supposed to act at the center point, C; for then, it can be shown that rotation of the section is caused not only by the couple of moment, P times the distance, $C\,E$, but also, in part, by the force, P, at C, which causes a uniform stress, $\frac{P}{t}$, on the section.* To find the rotation of the section, $\alpha' - \alpha = \Delta\,\alpha$, caused by the couple of moment, P times the distance, $D\,E$, regard the left section as fixed and the angle, α, of the voussoir as indefinitely small, which is equivalent to taking the limit of the result as α approaches zero.

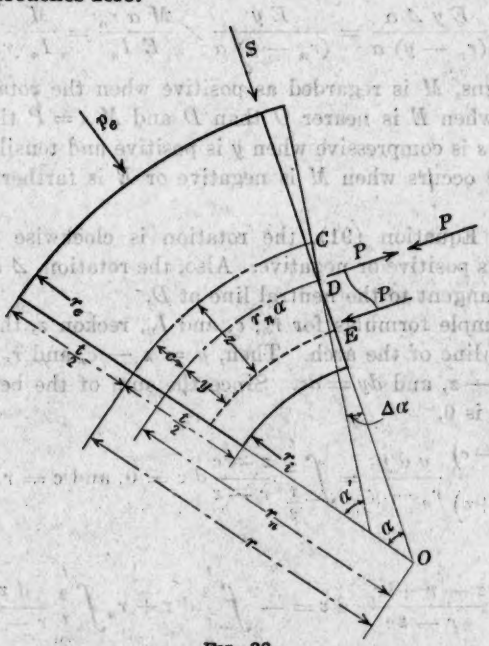

FIG. 28.

In Fig. 28, y, the distance of a fiber from the neutral axis, will be taken as positive or negative, according as the fiber lies nearer to or farther from O than the neutral axis. The length of the fiber is thus, $(r_n - y)\,\alpha$, and it is shortened by the rotation by an amount, $y\,\Delta\,\alpha$, so that its unit deformation or strain $= \frac{y\,\Delta\,\alpha}{(r_n - y)\,\alpha}$, and the unit stress, s, on this fiber $= E\,\frac{y\,\Delta\,\alpha}{(r_n - y)\,\alpha}$.

* See demonstration in connection with Fig. 33.

Regarding this fiber as having the area, $d\,y$ times 1, its stress will be, $E\,\frac{y\,d\,y\,\Delta\,\alpha}{(r_n - y)\,\alpha}$, and the sum of the moments of such stresses for the entire section must equal M.

Then,

$$M = \frac{E\,\Delta\,\alpha}{r_n\,\alpha}\int_{-\left(\frac{t}{2}+c\right)}^{\left(\frac{t}{2}-c\right)} \frac{y^2\,d\,y}{1 - \frac{y}{r_n}} = E\,\frac{\Delta\,\alpha}{r_n\,\alpha}\,I_n$$

in which,

$$I_n = \int_{-\left(\frac{t}{2}+c\right)}^{\left(\frac{t}{2}-c\right)} \frac{y^2\,d\,y}{1 - \frac{y}{r_n}} = r_n \int_{-\left(\frac{t}{2}+c\right)}^{\left(\frac{t}{2}-c\right)} \frac{y^2\,d\,y}{r_n - y} \quad \ldots\ldots\ldots\ldots (91)$$

Then,

$$\Delta\,\alpha = \frac{M\,r_n\,\alpha}{E\,I_n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (92)$$

and,

$$s = \frac{E\,y\,\Delta\,\alpha}{(r_n - y)\,\alpha} = \frac{E\,y}{(r_n - y)\,\alpha} \times \frac{M\,\alpha\,r_n}{E\,I_n} = \frac{M}{I_n}\,\frac{y\,r_n}{r_n - y} \quad \ldots\ldots\ldots\ldots (93)$$

As to the signs, M is regarded as positive when the rotation is clockwise, and, therefore, when E is nearer O than D and M ($= P$ times the distance, DE) is positive, s is compressive when y is positive and tensile when y is negative; the reverse occurs when M is negative or E is farther from the center, O, than D.

Again, from Equation (91), the rotation is clockwise or anti-clockwise according as M is positive or negative. Also, the rotation, $\Delta\,\alpha$, is evidently the rotation of the tangent to the neutral line at D.

To deduce simple formulas for r_n, c, and I_n, reckon z, the radial ordinate, from the medial line of the arch. Then, $y = z - c$, and $r_n = r - c$; therefore, $r_n - y = r - z$, and $dy = dz$. Since the sum of the bending stresses on the cross-section is 0,

$$\int_{-\left(\frac{t}{2}+c\right)}^{\left(\frac{t}{2}-c\right)} \frac{y\,d\,y}{r_n - y} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{z - c}{r - z}\,d\,z = 0, \text{ and } c = r - r_n$$

therefore,

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{z - r + r_n}{r - z}\,d\,z = -\int_{-\frac{t}{2}}^{\frac{t}{2}} d\,z + r_n \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{d\,z}{r - z} = 0$$

Now,

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{d\,z}{r - z} = -\log_e (r - z)\Big]_{-\frac{t}{2}}^{\frac{t}{2}} = \log_e \frac{r + \frac{t}{2}}{r - \frac{t}{2}} = \log_e \frac{r_e}{r_i}$$

therefore,

$$r_n \log_e \frac{r_e}{r_i} = t$$

or,

$$r_n = \frac{t}{\log_e\left(\frac{r_e}{r_i}\right)} \quad \ldots\ldots (94)^*$$

$$c = r - r_n = r - \frac{t}{\log_e\left(\frac{r_e}{r_i}\right)} \quad \ldots\ldots (95)$$

$$I_n = r_n \int_{-\left(\frac{t}{2}+c\right)}^{\left(\frac{t}{2}-c\right)} \frac{y^2\, dy}{r_n - y} = r_n \int_{-\left(\frac{t}{2}+c\right)}^{\left(\frac{t}{2}-c\right)} \left(r_n \frac{y\, dy}{r_n - y} - y\, dy\right)$$

$$= 0 - r_n \int_{-\left(\frac{t}{2}+c\right)}^{\left(\frac{t}{2}-c\right)} y\, dy = -r_n \int_{-\frac{t}{2}}^{\frac{t}{2}} (z-c)\, dz$$

Now, since z is measured from the center of gravity of the section,

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} z\, dz = 0$$

and,

$$I_n = c\, r_n\, t \quad \ldots\ldots (96)$$

The numerical computation of $\log_e \frac{r_e}{r_i} = \log_e r_e - \log_e r_i$, is quickly effected when an adequate table of Napierian logarithms is at hand. Otherwise, $\log_e N = 2.30258509 \times \log_{10} N$. By use of Equations (95) and (96), letting $I = \frac{1}{12} t^3 =$ moment of inertia of cross-section,

If $t = 30$ and $r = 100$, or $\frac{t}{r} = 0.3$, then $I_n = 0.99841\ I$

If $t = 50$ and $r = 100$, or $\frac{t}{r} = 0.5$, then $I_n = 0.99568\ I$

If $t = 70$ and $r = 100$, or $\frac{t}{r} = 0.7$, then $I_n = 0.99123\ I$

Thus, I_n differs from I by less than 1% for any case likely to occur in practice, and it will be replaced by I in the numerical computations.

Moments Taken About Points in Neutral Axis.—It will not be proved that the resultant, P, of the normal stresses on the cross-section that cause it to move to a parallel position, or that cause translation only, must pass through D.

* Expanding,

$$\log_e \frac{r_e}{r_i} = \log_e \frac{r + \frac{t}{2}}{r - \frac{t}{2}},$$

$$\log_e \frac{r_e}{r_i} = \frac{t}{r}\left\{1 + \frac{1}{12}\left(\frac{t}{r}\right)^2 + \frac{1}{80}\left(\frac{t}{r}\right)^4 + \frac{1}{448}\left(\frac{t}{r}\right)^6 + \ldots\ldots\right\}$$

If this is substituted in Equation (94), it is seen that $r_n < r$, or the neutral line, is nearer 0 than the medial line of the arch, as was assumed.

Let,

σ = one of these unit stresses, acting at a distance, r', from 0.

α = the angle between the two sections of the voussoir, Fig. 29.

The length of a fiber at $I = I\,I' = r'\,\alpha$. Since the unit stress acting on it equals σ, it will shorten the amount, $I\,I'' = \frac{\sigma}{E}\,r'\,\alpha$. Since by assumption, the shortening is to be the same at any point of the section, $I\,I' = A\,A'' = BB''$, therefore,

$$\sigma\,r' = \text{constant} = k \text{ (say)}$$

Then,

$$P = \int_{r_i}^{r_e} \sigma\,d\,r' = k\int_{r_i}^{r_e} \frac{d\,r'}{r'} = k \log_e \frac{r_e}{r_i}$$

or,

$$k = \frac{P}{\log_e\left(\frac{r_e}{r_i}\right)}$$

that is,

$$\sigma = \frac{k}{r'} = \frac{P}{r' \log_e\left(\frac{r_e}{r_i}\right)} = \frac{P}{t}\,\frac{r_n}{r'} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (97a)$$

Note that at D, $r' = r_n$, and $\sigma = \frac{P}{t}$.

The unit stresses, σ_e and σ_i, at the extrados and intrados, respectively, are:

$$\sigma_e = \frac{P}{r_e \log_e\left(\frac{r_e}{r_i}\right)};\quad \sigma_i = \frac{P}{r_i \log_e\left(\frac{r_e}{r_i}\right)} \qquad \ldots\ldots\ldots\ldots\ldots (97b)$$

The position of the resultant, P, of these stresses can be most conveniently found by taking moments about O. From Equation (97a),

$$\int_{r_2}^{r_e} \sigma\,r'\,d\,r' = \int_{r_2}^{r_e} k\,d\,r' = k\,(r_e - r_i) = k\,t = P\,r_n$$

Thus, P passes through D.

It follows, since P, or the forces, σ, cause translation only, that the couple of moment, P times the distance, $D\,E$ (Fig. 28), causes all the rotation. This moment is always equal to the moment of all forces supposedly acting from the crown to the section about Point D (and not C). To include all the rotation of sections, therefore, the bending moments must be taken about points on the neutral axis. For thin arches, such as those used on highways or railways, it makes no practical difference whether the moments, M, are taken about C or D, but for the thick arches of the lower parts of arched dams 100 to 400 ft. high, the difference is quite appreciable.

Elastic Work of Semi-Arch Due to P.—To find the elastic work of the voussoir of angle, $\alpha = d\,\phi$, note that the unit strain at y is, $\frac{\sigma\,r'\,d\,\phi}{E} = \frac{k\,d\,\phi}{E}$,

and that the work of the fiber of cross-section, $d\,y \times 1$, for a uniformly increasing load, is,

$$\frac{1}{2}\,\sigma\,(d\,r' \times 1)\,\frac{k\,d\,\phi}{E} = \frac{d\,\phi}{2\,E}\,k^2\,\frac{d\,r'}{r'},$$

since

$$\sigma = \frac{k}{r'}.$$

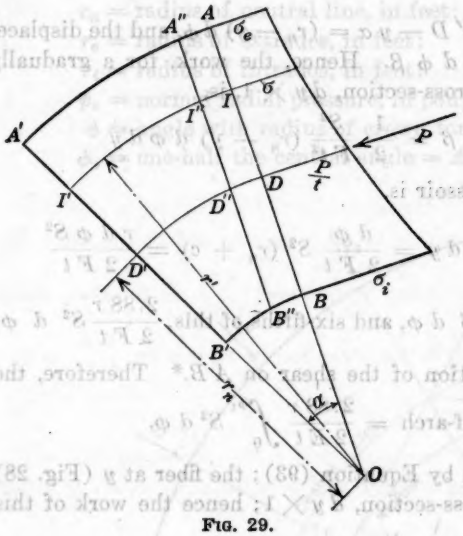

FIG. 29.

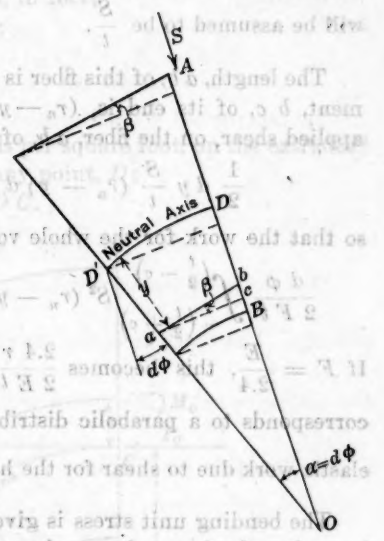

FIG. 30.

Thus, the work due to P acting at the point, D, of the voussoir is,

$$\frac{d\,\phi}{2\,E}\,k^2 \int_{r_i}^{r_e} \frac{d\,r'}{r'} = \frac{k^2\,d\,\phi}{2\,E} \log_e \frac{r_e}{r_i}$$

or, since,

$$k = \frac{P}{\log_e \left(\frac{r_e}{r_i}\right)}, \text{ and } \log_e \left(\frac{r_e}{r_i}\right) = \frac{t}{r_n}$$

the work is,

$$\frac{r_n}{2\,E\,t}\,P^2\,d\,\phi$$

The work for the semi-arch having a central angle $2\,\phi_1$ is,

$$\frac{r_n}{2\,E\,t} \int_0^{\phi_1} P^2\,d\,\phi\,.$$

This is the work due to translation only caused by the normal component, P.

The work done by the shearing component, S, can be found by aid of Fig. 30, in which, the central angle, $d\,\phi$, of the voussoir is infinitesimal, so that the limit, as $d\,\phi = 0$, is tacitly assumed in stating results. The shear is supposed

to displace the fibers relative to the section through D' to the positions shown by the dotted lines (really curved), the angle of rotation being $\beta = \frac{S}{t\ F}$, $t \times 1$ being the area of the section and F the shearing modulus. The shear is not uniformly distributed over the cross-section, $A\ B$; but assume for the present that it is, and that the correction will be eventually applied. Thus, for the fiber, $a\ b$, distant y from the neutral axis, the unit shear acting at b will be assumed to be $\frac{S}{t}$.

The length, $a\ b$, of this fiber is $D'\ D - y\ \alpha = (r_n - y)\ d\phi$, and the displacement, $b\ c$, of its end is, $(r_n - y)\ d\ \phi\ \beta$. Hence, the work, for a gradually applied shear, on the fiber, $a\ b$, of cross-section, $d\ y \times 1$, is,

$$\frac{1}{2}\ d\ y\ \frac{S}{t}\ (r_n - y)\ d\ \phi\ \beta = \frac{1}{2}\ \frac{S^2}{F\ t^2}\ (r_n - y)\ d\ \phi\ d\ y$$

so that the work for the whole voussoir is,

$$\frac{d\ \phi}{2\ F\ t^2} \int_{-\left(\frac{t}{2}+c\right)}^{\left(\frac{t}{2}-c\right)} S^2\ (r_n - y)\ d\ y = \frac{d\ \phi}{2\ F\ t}\ S^2\ (r_n + c) = \frac{r\ d\ \phi\ S^2}{2\ F\ t}$$

If $F = \frac{E}{2.4}$, this becomes $\frac{2.4\ r}{2\ E\ t}\ S\ d\ \phi$, and six-fifths of this, $\frac{2.88\ r}{2\ F\ t} S^2\ d\ \phi$, corresponds to a parabolic distribution of the shear on $A\ B$.* Therefore, the elastic work due to shear for the half-arch $= \frac{2.88\ r}{2\ E\ t} \int_0^{\phi_1} S^2\ d\ \phi$.

The bending unit stress is given by Equation (93); the fiber at y (Fig. 28) has a length, $(r_n - y)\ \alpha$, and a cross-section, $d\ y \times 1$; hence the work of this fiber is,

$$\frac{1}{2}\ s\ d\ y\ \frac{(r_n - y)\ \alpha\ s}{E} = \frac{M^2}{2\ E\ {I_n}^2}\ \frac{{r_n}^2\ y^2}{(r_n - y)^2}\ (r_n - y)\ \alpha\ d\ y = \frac{{r_n}^2\ \alpha}{2\ E\ {I_n}^2}\ \frac{M^2\ y^2\ d\ y}{r_n - y}$$

By Equation (91),

$$\int_{-\left(\frac{t}{2}+c\right)}^{\left(\frac{t}{2}-c\right)} \frac{y^2\ d\ y}{r_n - y} = \frac{I_n}{r_n}$$

and, therefore, the elastic work of the voussoir of angle, $d\ \phi$, due to M (taken about D) is

$$\frac{r_n}{2\ E\ I_n}\ M^2\ d\ \phi$$

Then, the work for the semi-arch of the central angle, $2\ \phi$, due to M is,

$$\frac{r_n}{2\ E\ I_n} \int_0^{\phi_1} M^2\ d\ \phi.$$

Hence, the elastic work for the semi-arch due to M, P, and S is,

$$L = \frac{r_n}{2\ E\ I_n} \int_0^{\phi_1} M^2\ d\ \phi + \frac{r_n}{2\ E\ t} \int_0^{\phi_1} P^2\ d\ \phi + \frac{2.88\ r}{2\ E\ t} \int_0^{\phi_1} S^2\ d\ \phi. \quad (98)\dagger$$

* This is, of course, only approximate, and is only true when $r_n = r$.

† In this formula for L, only the work of the radial stress is omitted. Its influence is very small, as will appear subsequently.

Solution of the Arch.—Thrusts and Moments.—The arch to be considered is a horizontal circular arch of constant radial thickness, t, and a vertical height of 1 ft., fixed at the ends, and subjected on the extrados to a water pressure of p_e lb. per sq. ft. Generally, this is not the full water pressure at the level of the medial plane of the arch, but the part carried by the arch, the remainder being carried by the supposed cantilever.

In Fig. 31, representing the half-arch,

t = uniform radial thickness of arch, in feet;
r = radius of the center line, in feet;
r_n = radius of neutral line, in feet;
r_e = radius of extrados, in feet;
r_i = radius of intrados, in feet;
p_e = normal radial pressure, in pounds per square foot, on the extrados;
ϕ = angle with radius of crown for any point, D;
ϕ_1 = one-half the central angle = $A\ O\ C$.

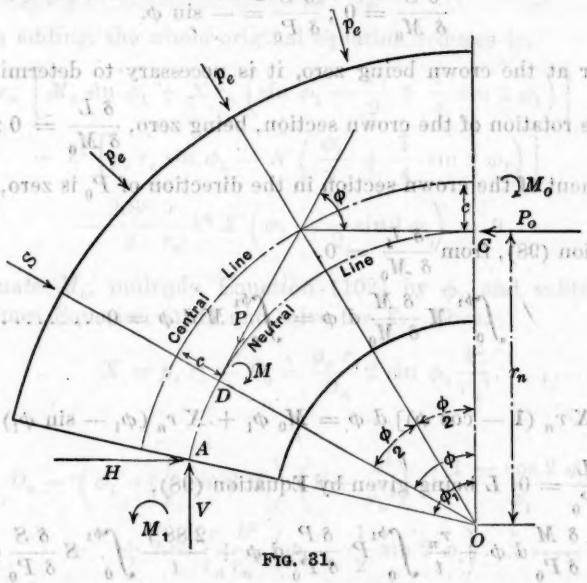

FIG. 31.

The arc, $A\ D\ C$, is the neutral line, so that if s = the length of the arc, the distance, $C\ D = r_n\ \phi$, $d\ s = r_n\ d\ \phi$.

P_0 = thrust at the crown, acting at C, on the neutral line;
M_0 = moment at crown, taken positive when clockwise;
M, P, and S = respectively, the moment, normal component, and shear on the section through D $(r_n\ \phi)$;
$c = r - r_n$;
η = radial deflection at the crown; and
E = modulus of elasticity of the arch.

At the abutments, the components of the reaction parallel to and perpendicular to $O\ C$ are V and H and the reaction moment is M_1. For the semi-

arch of Fig. 31, the reactions, V, H, M_1, at the left abutment, with P_0, M_0, at the crown, and the water load on the semi-arch, hold the latter in equilibrium, so that it can be treated as a free body.

The analysis now proceeds exactly according to the method used by the writer in his paper on "The Circular Arch Under Normal Loads"*. The water pressure, however, is no longer referred to the center line of the arch ring, but to the extrados, and the cylinder formula is not used. Then, in the paper† cited, letting, $X = p_e r_e - P_0$,

$$\left.\begin{aligned} P &= p_e r_e - X \cos \phi \\ M &= M_0 + X r_n (1 - \cos \phi) \\ S &= X \sin \phi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (99)$$

Taking partial derivations of these equations,

$$\frac{\delta M}{\delta M_0} = 1, \frac{\delta M}{\delta P_0} = -r_n (1 - \cos \phi); \frac{\delta P}{\delta M_0} = 0, \frac{\delta P}{\delta P_0} = \cos \phi;$$

$$\frac{\delta S}{\delta M_0} = 0; \frac{\delta S}{\delta P_0} = -\sin \phi.$$

The shear at the crown being zero, it is necessary to determine only M_0, and P_0. The rotation of the crown section, being zero, $\frac{\delta L}{\delta M_0} = 0$; also, since the displacement of the crown section in the direction of P_0 is zero, $\frac{\delta L}{\delta P_0} = 0$.

By Equation (98), from $\frac{\delta L_i}{\delta M_0} = 0$,

$$\int_0^{\phi_1} M \frac{\delta M}{\delta M_0} d\phi = \int_0^{\phi_1} M d\phi = 0 \ldots\ldots\ldots\ldots\ldots (100)$$

therefore,

$$\int_0^{\phi_1} [M_0 + X r_n (1 - \cos \phi)] d\phi = M_0 \phi_1 + X r_n (\phi_1 - \sin \phi_1) = 0 \ldots (101)$$

From, $\frac{\delta L}{\delta P_0} = 0$, L being given by Equation (98),

$$\frac{r_n}{I_n} \int_0^{\phi_1} M \frac{\delta M}{\delta P_0} d\phi + \frac{r_n}{t} \int_0^{\phi_1} P \frac{\delta P}{\delta P_0} d\phi + \frac{2.88\, r}{t} \int_0^{\phi_1} S \frac{\delta S}{\delta P_0} d\phi = 0$$

Multiplying by $\frac{I_n}{r_n}$ and putting $\frac{I_n}{t} = k^2$,

$$-r_n \int_0^{\phi_1} M (1 - \cos \phi) d\phi + k^2 \int_0^{\phi_1} P \cos \phi \, d\phi + \frac{2.88\, r}{r_n} k^2 \int_0^{\phi_1} S (-\sin \phi) d\phi = 0$$

By Equation (100),

$$\int_0^{\phi_1} M d\phi = 0$$

* *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), pp. 236–237.
† *Loc. cit.*, p. 236.

hence, the first term reduces to,

$$r_n \int_0^{\phi_1} M \cos\phi \, d\phi = r_n \int_0^{\phi_1} [M_0 + X r_n (1 - \cos\phi)] \cos\phi \, d\phi$$

$$= r_n \left[M_0 \sin\phi_1 + X r_n \left(\sin\phi_1 - \frac{\phi_1}{2} - \frac{1}{4} \sin 2\phi_1 \right) \right]$$

Likewise,

$$k^2 \int_0^{\phi_1} P \cos\phi \, d\phi = k^2 \int_0^{\phi_1} (p_e r_e - X \cos\phi) \cos\phi \, d\phi$$

$$= k^2 \left[p_e r_e \sin\phi_1 - X \left(\frac{\phi_1}{2} + \frac{1}{4} \sin 2\phi_1 \right) \right]$$

Finally, the term,

$$-2.88 \frac{r k^2}{r_n} \int_0^{\phi} S \sin\phi \, d\phi = -\frac{2.88}{2} \frac{r}{r_n} k^2 X \left(\phi_1 - \frac{1}{2} \sin 2\phi_1 \right)$$

Hence, on adding, the whole original equation reduces to,

$$r_n \left[M_0 \sin\phi_1 + X r_n \left(\sin\phi_1 - \frac{\phi_1}{2} + \frac{1}{4} \sin 2\phi_1 \right) \right]$$

$$+ k^2 \left[p_e r_e \sin\phi_1 - X \left(\frac{\phi_1}{2} + \frac{1}{4} \sin 2\phi_1 \right) \right]$$

$$- \frac{2.88}{2} \frac{r}{r_n} k^2 X \left(\phi_1 - \frac{1}{2} \sin 2\phi_1 \right) = 0 \quad \ldots\ldots\ldots\ldots (102)$$

To eliminate M_0, multiply Equation (102) by ϕ_1 and subtract it from ($r_n \sin\phi_1$) times Equation (101) and solve for X. Then,

$$X = p_e r_e - P_0 = \frac{p_e r_e}{D_n} 2 \sin\phi_1 \frac{k^2}{r_n^2} \quad \ldots\ldots\ldots\ldots (103)^*$$

in which,

$$D_n = \left(\phi_1 + \frac{1}{2} \sin 2\phi_1 \right) \left(1 + \frac{k^2}{r_n^2} \right) - \frac{1 - \cos 2\phi_1}{\phi_1}$$

$$+ 2.88 \frac{r}{r_n} \frac{k^2}{r_n^2} \left(\phi_1 - \frac{1}{2} \sin 2\phi_1 \right) \quad \ldots\ldots\ldots\ldots (104)$$

---

* To compare with the author's Equation (30), the water pressure must be referred to the extrados so that in the formula for deflection of the full cylinder, $e = \frac{p r \cdot r}{E t}$, $p r$ must be replaced by $p_e r_e$. Then, as given by the author,

$$e' = e \frac{r_n}{r} = \frac{p_e r_e \cdot r_n}{E t}$$

which agrees with Equation (38*b*) p. 219. Therefore, Equation (30) p. 217, becomes,

$$X' = \frac{p_e r_e}{E t} \frac{r_n \sin\phi_1}{y'}$$

In Equation (28) p. 217, r_n should be replaced by r, according to the usual theory. With this correction, the value of X' can be easily reduced to Equation (103). The numerical value of the term in which this error occurs is very small and the error is negligible. In fact, it is perhaps well to replace $\frac{r}{r_n}$ by unity in the last term of Equation (104). It is gratifying to have this complete check of the fundamental formula for X by a totally different method.

From Equation (101):

$$M_0 = -X r_n \left(1 - \frac{\sin \phi_1}{\phi_1}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (105)$$

This moment at the crown is always negative or counterclockwise, giving tension at the intrados and compression at the extrados.

If this value of M_0 is substituted in Equation (99), the value of M at any point, (r_n, ϕ), is found, as follows:

$$M = +X r_n \left(\frac{\sin \phi_1}{\phi} - \cos \phi\right) \ldots\ldots\ldots\ldots\ldots\ldots (106)$$

This moment for a point, D' (Fig. 32), is equal to $-X$ times the distance, $E'D'$, and for a point, D'', it equals $+X$ times the distance, $D''E''$, if X is laid off as a force, acting to the right at E on OC, where $OE = r_n \frac{\sin \phi_1}{\phi_1}$. The moment is zero at D_0, where the line of action of X crosses the neutral line; it is positive for points between D_0 and A and negative for points on the neutral axis between D_0 and C.

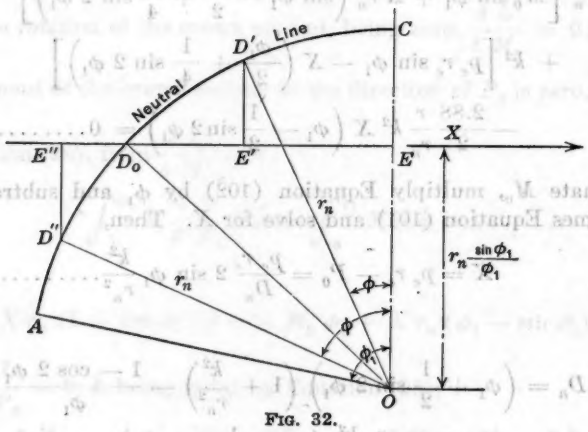

FIG. 32.

At the abutment, A, $\phi = \phi_1$ in Equation (106), and since the moment is positive or clockwise, it causes tension at the extrados of the abutment section and compression at the intrados.

In Equations (103) and (104), $k^2 = \frac{I_n}{t}$. As already noted, I_n differs from I by less than 1% for most cases in practice. Assuming,

$$k^2 = \frac{I}{t} = \frac{1}{12} t^2,$$

and, since by Equation (94),

$$r_n = \frac{t}{\log_e \left(\frac{r_e}{r_i}\right)}$$

therefore,

$$\frac{k^2}{r_n^2} = \frac{1}{12}\left(\log_e \frac{r_e}{r_i}\right)^2.$$

In numerical computations, it is well to use a seven-place logarithmic table for the first two terms of D_n, as their difference for small angles is very small.

Example.—The unit being the foot, let $t = 30$, $r = 100$, $2\phi_1 = 40°$. Then,

$$r_e = 115; \quad r_i = 85; \quad \log_e\left(\frac{r_e}{r_i}\right) = \log_e 115 - \log_e 85 = 0.302281$$

$$r_n = \frac{30.000}{0.302281} = 99.2455; \quad c = 0.7545; \quad \frac{k^2}{r^2} = \frac{(0.3023)^2}{12} = 0.007615$$

$$\frac{r}{r_n} = 1.00760; \quad 2.88\,\frac{r}{r_n}\,\frac{k^2}{r_n^2} = 0.022098$$

From Equations (101) and (102),

$$D_n = 0.6755655 - 0.6702333 + 0.0006115 = 0.0059437$$

$$X = \frac{115\,p_e \times 2 \times 0.007615 \sin 20°}{0.0059437} = 100.78\,p_e$$

$$P_0 = p_e r_e - X = (115 - 100.78)\,p_e = 14.22\,p_e$$

At the abutment,

$$P_1 = p_e r_e - X\cos\phi_1 = (115 - 100.78 \times \cos 20°)\,p_e = 20.29\,p_e.$$

From Equations (105) and (106),

$$M_0 = -100.78 \times 99.2455\,(1 - 0.979815)\,p_e = -202\,p_e,$$

$$M_1 = 100.78 \times 99.2455\,(0.979815 - 0.939693)\,p_e = 401.3\,p_e.$$

Unit Tangential Stresses.—The unit stresses at the extrados and the intrados, at the crown and the abutment will now be computed. Tension will be taken as positive and compression as negative, except that p_e will always be treated as positive. This agrees with the author's conventions. If p_e is expressed in pounds per square foot, then the stresses will be in pounds per square foot, and it will be necessary to divide by 144 to reduce to pounds per square inch. This is equivalent to expressing p_e in pounds per square inch, and this will be supposed in the formulas, so that the total unit stresses given by the formulas will be in pounds per square inch.

The bending stresses are obtained from Equation (93) by putting $y = \frac{t}{2} - c$ for the intrados, $y = -\left(\frac{t}{2} + c\right)$ for the extrados, and assuming $I_n = I = \frac{1}{12}t^3$. They are represented by the first terms in the formulas, the signs being assigned by the conventions stated.

The stresses due to the component, P, normal to the section and acting where the neutral line crosses it, are given by Equation (97*a*), with the negative sign, since these stresses are all compressive.

Let s_e, s_i, equal the total unit stress on a fiber at the extrados and intrados, respectively.

At the crown,

$$\left.\begin{aligned} s_e &= \frac{M_0}{I}\,\frac{\frac{t}{2}+c}{r_e}\,r_n - \frac{P_0}{r_e \log_e\left(\frac{r_e}{r_i}\right)} \\ s_i &= -\frac{M_0}{I}\,\frac{\frac{t}{2}-c}{r_i}\,r_n - \frac{P_0}{r_i \log_e\left(\frac{r_e}{r_i}\right)} \end{aligned}\right\} \quad \text{(107)}$$

At the abutment,

$$\left.\begin{aligned} s_e &= \frac{M_1}{I}\,\frac{\frac{t}{2}+c}{r^e}\,r_n - \frac{P_1}{r_e \log_e\left(\frac{r_e}{r_i}\right)} \\ s_i &= -\frac{M_1}{I}\,\frac{\frac{t}{2}-c}{r_i}\,r_n - \frac{P_1}{r_i \log_e\left(\frac{r_e}{r_i}\right)} \end{aligned}\right\}$$

On substituting numerical values, always giving M_0 its negative sign:* At the crown,

$$s_e = (-1.221 - 0.409)\ p_e = -1.630\ p_e \text{ (compression)}$$

$$s_i = (+1.493 - 0.553)\ p_e = +0.940\ p_e \text{ (tension)}$$

At the abutment,

$$s_e = (+2.425 - 0.584)\ p_e = +1.841\ p_e \text{ (tension)}$$

$$s_i = (-2.967 - 0.789)\ p_e = -3.756\ p_e \text{ (compression)}$$

Since the author's value of X', as given by Equation (30),† with the negligible correction, can be reduced to the writer's Equation (103), the values of s_e and s_i, as computed by either method, should give identical results. Thus far, the check is complete.

Equations (107) can be put in another form. Thus, in Fig. 28, E being a point on the equilibrium polygon, suppose the two opposed forces, P, to be placed at C, the center of the section, in place of at D on the neutral axis. Then the single force, P, at C is the resultant of the uniform stress, $\frac{P}{t}$, acting on the section, and the couple gives the moment,

$M_c = P$ times the distance, $E\,C = P$ (distance, $E\,D$ + distance, $D\,C$)

$$= M_D + Pc$$

in which, $M_D = M$ of Equations (107) equals the moment of all forces to the right of the section about D, a point on the neutral axis. The same formula

* Here, $\frac{\frac{t}{2}+c}{Ir_e} = 0.00604$; $\frac{\frac{t}{2}-c}{Ir_i} = 0.00740$; $r_e \log_e \frac{r_e}{r_i} = 34.73$; $r_i \log_e \frac{r_e}{r_i} = 25.27$.

† See p. 217.

holds when E is farther from O than D, the sign of M then being minus, but the sign of $P\,c$ is always plus.

The formulas for s_e and s_i, for any section the normal component of which is P and the moment about D, a point on the neutral axis, is M, become:

$$\left.\begin{aligned} s_e &= \frac{(M + P\,c)}{I}\,\frac{\frac{t}{2} + c}{r_e}\,r_n - \frac{P}{t} \\ s_i &= -\frac{(M + P\,c)}{I}\,\frac{\frac{t}{2} - c}{r_i}\,r_n - \frac{P}{t} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (108)$$

At the crown, $M = M_0$, $P = P_0$, and at the abutment, $M = M_1$, $P = P_1$. Thus, at the crown, $M = -202\,p_e$, $P_0 = 14.22\,p_e$, and $c = 0.7545$. Then, $(M + P\,c) = -191.28\,p_e$ and by Equations (108),

$$s_e = \left[-191.28 \times 0.0060426 - \frac{14.22}{30}\right] p_e = -1.630\,p_e$$

exactly as found previously; and, similarly, for other cases.

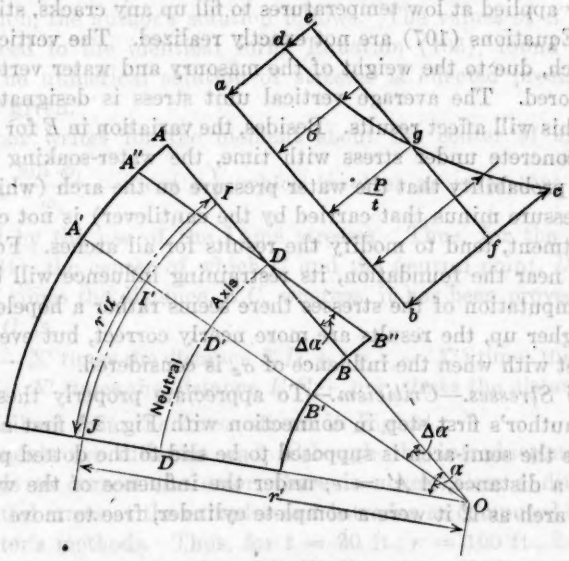

FIG. 33.

The effect of the uniform compression, $\frac{P}{t}$, represented by the ordinates of the rectangle, $eabf$, of Fig. 33, on the voussoir of the central angle, α, may now be examined.

The forces, $\frac{P}{t}$, act on the face, $A\ B$, and rotate it to the position, $A'\ B'$, since any fiber, $I\ J$, the length of which is $r'\,\alpha$, is shortened by the distance,

$I\,I' = \frac{P}{t}\,\frac{r'\,\alpha}{E}$, which varies with r' and, thus, $A'\,I'\,B'$ is a straight line. The angle of rotation is, $\Delta\,\alpha = \frac{I\,I'}{r'} = \frac{P}{t}\,\frac{\alpha}{E}$.

Now suppose the forces in the areas, *edg* and *gfc*, form a couple of moment, m, that causes $A\,B$ to rotate to $A''\,B''$ about D through the angle $\Delta\,\alpha$; then the forces, σ, in the area, *gdabc*, cause the translation from $A''\,B''$ to $A'\,B'$ and are given by Equation (95).

By Equation (92), $\Delta\,\alpha = \frac{m\,r_n\,\alpha}{E\,I_n}$, and this equals $\frac{P}{t}\,\frac{\alpha}{E}$. Therefore, $m = \frac{P\,I_n}{r_n\,t} = P\,c$, by Equation (94), or since the section, $A\,B$, turns counterclockwise, $m = -P\,c$. This moment combined with the moment, $M + P\,c$, of Equations (108), gives M, the moment about D, on the neutral axis, as the moment corresponding to the entire bending. Thus is proved this important principle in another way. Equations (108) are now transformed to Equations (107).

Suppose that the pressure grouting, which the author properly insists upon, to be properly applied at low temperatures to fill up any cracks, still the stresses as given by Equations (107) are not exactly realized. The vertical stress on a horizontal arch, due to the weight of the masonry and water vertically over it, has been ignored. The average vertical unit stress is designated σ_x by the author, and this will affect results. Besides, the variation in E for large stresses, the flow of concrete under stress with time, the water-soaking of the upper face, and the probability that the water pressure on the arch (which equals the full water pressure minus that carried by the cantilever) is not constant from crown to abutment, tend to modify the results for all arches. For those horizontal arches near the foundation, its restraining influence will be so marked that exact computation of the stresses there seems rather a hopeless task. For the arches higher up, the results are more nearly correct, but even there, difficulties are met with when the influence of σ_x is considered.

The Lamé Stresses.—Criticism.—To appreciate properly these difficulties, consider the author's first step in connection with Fig. 2,* first assuming that $\sigma_x = 0$, where the semi-arch is supposed to be slid to the dotted position along the skewback a distance, $A\,A' = e$, under the influence of the water pressure acting on the arch as if it were a complete cylinder, free to move and not fixed at the ends.

In this first stage, the stresses induced are the Lamé stresses, both tangential and radial. In deducing these stresses it was assumed that radial sections remain plane under the stresses, so that the assumed sliding can be effected. However, it will conduce to simplicity if the Lamé tangential stresses are considered as decomposed into a uniform stress and a couple. Thus, if R equals the resultant of the Lamé stresses, acting a distance, l, from the center of the section, on adding the forces, $+R$ and $-R$, at this center, there results the single compressive force, R, at the center, and a couple which can be resolved

* See p. 211.

into a uniform stress, $\frac{R}{t}$, and a couple of moment, $R\,l$. The latter affects the bending only and will be necessarily included in the moment of X' about the point where the neutral axis cuts the section, since the writer has demonstrated that all bending corresponds to such moments.* The only stresses then to be considered in this sliding stage are the uniform stresses, $\frac{R}{t}$, for which plane sections remain plane after strain, so that the sliding can be effected. To find the value of R, write the Lame unit stress in the form, $\frac{b}{x^2} + a$, in which, x = radius to the point on the section, and, $a = \frac{p_e r_e^2}{r_e^2 - r_i^2}$, and $b = \frac{p_e r_e^2 r_i^2}{r_e^2 - r_i^2}$. Then, by a simple integration and reduction,

$$R = \int_{r_i}^{r_e} \left(\frac{b}{x^2} + a\right) d\,x = p_e\,r_e$$

With the corresponding value of the constant unit stress, $\left(-\frac{p_e\,r_e}{t}\right)$ on each section, the author's solution follows. The values of X' thus deduced can be reduced to the identical form, Equation (103), found by the "work method". The numerical solution of stresses is effected precisely as in the last example given.

The author writes for the moment about the center of any section, C, $M_c = X'\left(r_n \frac{\sin \phi_1}{\phi_1} - r\cos\phi\right)$, which involves an error that is only partly compensated by the use of the Lamé stresses. Thus, for the crown section, $A\,B$ (Fig. 34), the center of which C and the neutral point, D, this becomes $M_c = -X'$ times the distance, $E\,C$, whereas, it has been proved that the true moment at C is,

$M + P_c = -X'$ times the distance $E\,D + (p_e\,r_e - X')$ times the distance $D\,C$
$= -X'$ times the distance $E\,C + p_e\,r_e$ times the distance $D\,C$

which also follows from the forces shown in Fig. 34.

The moments are different and, although the introduction of the Lamé stresses tends to correct the error, there is nevertheless a difference in the total tangential stress at the extrados and intrados as computed by the author's and the writer's methods. Thus, for $t = 30$ ft., $r = 100$ ft., $2\,\phi_1 = 40°$, and $m = \infty$, at the crown, by the writer's Equations (108) it was found that $s_e = -1.63\ p_e$, and $s_i = +0.94\ p_e$; whereas by the author's formulas, $s_e = -1.73\ p_e$, and $s_i = +1.01\ p_e$.

For the same arch, only assuming $m = 8$, $\sigma_x = 4.32\ p_e$, the writer's Equations (108) gives, $s_e = -1.91\ p_e$; $s_i = +0.04\ p_e$, which may be compared with the author's $s_e = -2.00\ p_e$; $s_i = +0.12\ p_e$.

For the very thick arch, $t = 100$ ft., $r = 100$ ft., $m = \infty$, and $2\,\phi_1 = 40°$, the writer's Equations (107) gives at the crown, $s_e = -0.198\ p_e$, $s_i = -0.051\ p_e$

* Hence, bending stresses due to this couple of moment, $R\,l$, should be neglected in finding the total tangential stresses.

—both compression—whereas from the author's formulas, $s_e = -0.527\ p_e$ (compression) and $s_i = +0.408\ p_e$ (tension)!

The error by the author's formulas is too great. Part of it is due to the use of the Lamé stresses involving, as just shown, bending stresses which should be neglected, since the moment of X' takes care of all bending stresses. By the writer's method the Lamé stresses are not introduced at all and Equations (107) or Equations (108) will give accurately the tangential stresses. The examples solved will illustrate the procedure.

The writer regrets to have to criticize this excellent paper, but he is constrained to regard the author's formulas for tangential stress as approximate. Some idea of the amount of error can be gained from a consideration of the numerical results just given.

Influence of the Unit Circumferential Strain.—The true unit circumferential strain, ϵ', at the neutral axis for the full cylinder, under the water load, p_e lb. per sq. ft., is,

$$\epsilon' = \frac{\sigma_t''}{E} - \frac{\sigma_r''}{E} - \frac{\sigma_x}{E},$$

in which, σ_t'', σ_r'', and σ_x, denote the tangential, radial, and vertical unit stresses, respectively, at the neutral axis. These stresses, being compressive, are to be given the minus sign. By aid of Equations (36) and (37),* the expressions for Equations (38) and (38*a*)† follow immediately.

It will be observed that these formulas include the effect of the radial as well as of the vertical stress, omitted in the solution by the "work method", which thus corresponds to $m = \infty$, or Poisson's ratio equal to zero.

For $m = \infty$ and $\frac{t}{r} = 0.3$, Equation (38*a*) gives, $\epsilon' = -3.821\ \frac{p_e}{E}$. Now, if in Equation (38*a*), after developing, the terms in $\frac{t^2}{r^2}$ and $\frac{t^4}{r^4}$ be omitted, and C be put equal to 0, the formula reduces to Equation (38 *b*), or $\epsilon'' = -\frac{p_e\, r_e}{E\, t}$, which, for $\frac{t}{r} = 0.3$, gives, $\epsilon'' = 3.833\ \frac{p_e}{E}$, or practically the same as ϵ'. Thus, when $m = \infty$, the relative thickness of the arch affects very little the strain ϵ' Note that this value of ϵ'' is exactly the same as that used previously in checking Equation (103).

The influence of m is more pronounced. Thus, when $\sigma_x = 0$, for $m = 8$, let ϵ' be derived from Equation (38 *a*), and for $m = \infty$, let ϵ'' be computed from Equation (38 *b*), and let X' and X'' be the corresponding values of X. Then,

$$X' = -r_n\, \epsilon'\, \frac{\sin \phi_1}{y'};\quad X'' = -r_n\, \epsilon''\, \frac{\sin \phi_1}{y'},$$

or,

$$\frac{X''}{X'} = \frac{\epsilon''}{\epsilon'}.$$

* See pp. 218 and 219.

† See p. 219.

Thus, for,

$$\frac{t}{r} = 0, \frac{X''}{X'} = 1.000$$

$$\frac{t}{r} = 0.3, \frac{X''}{X'} = \frac{3.833}{3.747} = 1.023$$

$$\frac{t}{r} = 0.5, \frac{X''}{X'} = \frac{2.500}{2.399} = 1.042$$

Thus, X'' exceeds X' by only 2.3% for $\frac{t}{r} = 0.3$, and by 4.2% for $\frac{t}{r} = 0.5$. The values of X'' are exactly equal to those given by Equation (103), as derived by the "work method," and the excess of X'' over the exact value X', is almost entirely due to neglecting the elastic work of the radial stress in applying the work method. These numerical illustrations will give a good idea of the error involved which is generally negligible, so that Equation (103) can be regarded as a good practical formula for finding X when σ_x is ignored. The influence of σ_x is to lessen numerically ϵ' and thus to increase the percentages just found.

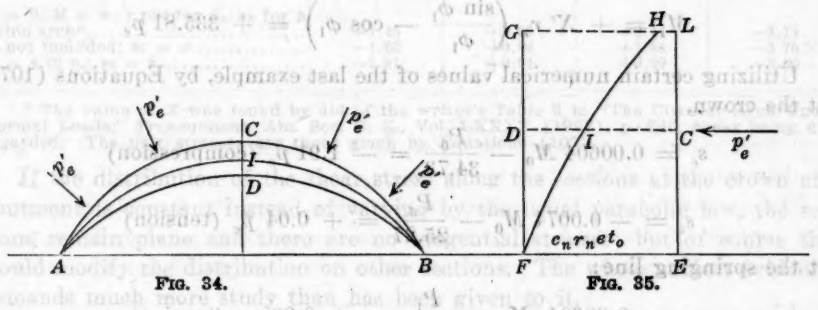

FIG. 34.

FIG. 35.

Influence of Vertical Load and of Poisson's Ratio.—Next, consider the case where σ_x is not zero and Poisson's ratio is introduced. The displacement for the free cylinder (Fig. 2), is $A\,A' = e' = -r_n\,\epsilon'$, in which, ϵ' is given by Equations (38) or (38*a*). The Lamé tangential stresses are as before, since their derivation, as given in textbooks, shows that they are independent of ϵ'; hence, reasoning as before for the sliding stage, each section will sustain a uniform compression of $p_e\,r_e$ lb., the sum of the Lamé stresses. Thus,

$$e = A\,A' = -r_n\,\varepsilon';\ X' = \frac{e' \sin \phi_1}{y'}.$$

If, now, bending stresses are computed from moments taken about points on the center line of the arch to get the total unit tangential stress, it is necessary, for any section, to add the uniform unit stress,

$$\frac{X' \cos \phi}{t} - \frac{p_e\,r_e}{t},$$

the first term being positive (tension) and the second, negative (compression).

Example.—The author, for an arch 400 ft. below the water surface, assumes that $\sigma_x = 300$ lb. per sq. in. and $p_e = 69.5$ lb. per sq. in. Then, $\sigma_x = 4.32\,p_e$.

For the arch previously considered ($t = 30$ ft., $r = 100$ ft., $2\phi_1 = 40°$), assume that $m = 8$ and $\sigma_x = 4.32\ p_e$; and, for the present, let p_e, σ_x, and E be expressed in pounds per square foot. Then, Equation (38) gives,

$$E\epsilon' = -3.206\ p_e$$

and Equations (29a) and (30) give:

$$y' = \frac{r_n^3}{E k^2 t}\frac{D_n}{2} = 0.0029718\ \frac{r_n^3}{E k^2 t}$$

and,

$$X' = \frac{-r_n \varepsilon' \sin\phi_1}{y'} = \frac{3.206 \sin\phi_1}{0.0029718}\frac{k^2}{r_n^2} t\, p_e = 84.29\ p_e.$$

As the writer's Equations (107) and (108) will be used in what follows, P_0 and P_1 will be given the plus sign. Therefore,

$$P_0 = p_e r_e - X' = (115 - 84.29)\ p_e = 30.71\ p_e$$

$$P_1 = p_e r_e - X' \cos\phi_1 = 35.97\ p_e$$

$$M_0 = -X' r_n \left(1 - \frac{\sin\phi_1}{\phi_1}\right) = -168.94\ p_e$$

$$M_1 = +X' r_n \left(\frac{\sin\phi_1}{\phi_1} - \cos\phi_1\right) = +335.81\ p_e$$

Utilizing certain numerical values of the last example, by Equations (107): At the crown,

$$s_e = 0.00604\ M_0 - \frac{P_0}{34.73} = -1.91\ p_e \text{ (compression)}$$

$$s_i = -0.0074\ M_0 - \frac{P_0}{25.27} = +0.04\ p_e \text{ (tension)}$$

At the springing line:

$$s_e = 0.00604\ M_1 - \frac{P_1}{34.73} = +0.99\ p_e \text{ (tension)}$$

$$s_i = -0.0074\ M_1 - \frac{P_1}{25.27} = -3.90\ p_e \text{ (compression)}$$

To check these values, use Equations (108), writing $c = 0.754$ ft.: At the crown,

$$M_0 + P_0 c = -145.8\ p_e;$$

At the springing line,

$$M_1 + P_1 c = +363.0\ p_e;$$

the moments being taken about the center of the sections. By substituting in Equations (108), the same values as before are derived.

Summary of Numerical Results.—The results are in pounds per square foot. On dividing by 144 the stress will be given in pounds per square inch. This amounts to expressing p_e in pounds per square inch in the final results. A summary of the numerical results are given in Table 10.

Influence of Shear on Tangential Stresses.—The influence of shear is not well understood, and in dealing with it, certain inconsistencies appear. It is known that, ordinarily, shear changes a plane section to a warped surface,

and the amount of the departure affects the computation of the tangential stress due to shear. However, at the crown, from considerations of symmetry, the section must remain plane; similarly at the abutment the section, being fixed by assumption, must remain plane. At these two sections the usual tangential stresses due to shear do not seem to be realized. Following the general lines of the analysis of J. P. J. Williams, M. Am. Soc. C. E.,* a rather cumbersome formula has been derived for the change in tangential stress for the arch due to shear. This may apply to intermediate sections, not too near the crown or the abutment, but it is not given because it cannot apply to the sections at the crown or abutment—the very ones to be investigated. The same reasoning applies to the center of a beam uniformly loaded, where the section must remain plane after the application of the load.

TABLE 10.—COMPARISON OF UNIT STRESSES AS FACTORS OF p_e

Assumptions.	Crown Section.		Springing Line.	
	Extrados.	Intrados.	Extrados.	Intrados.
$\sigma_x = 0$; $M = \infty$; r put for r_n as for a thin arch*	−1.45	+1.56	+2.44	−3.71
σ_x not included; $m = \infty$	−1.63	+0.94	+1.84	−3.76
$\sigma_x = 4.32\ p_e$; $m = 8$	−1.91	+0.04	+0.99	−3.90

* The value of X was found by aid of the writer's Table 6 in "The Circular Arch Under Normal Loads," *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 266, shear being disregarded. The unit stresses are those given by Equations (107).

If the distribution of the shear stress along the sections at the crown and abutment is constant instead of varying by the usual parabolic law, the sections remain plane and there are no tangential stresses; but of course this would modify the distribution on other sections. The whole subject evidently demands much more study than has been given to it.

Radial Deflection at the Neutral Axis at the Crown.—Fixed Ends.—The formula for deflection is derived exactly as given in the writer's previous paper,† using Equation (98) for L, the elastic work of deformation, and Equation (99) for the values of M, P, and S. Thus, the value of $\frac{E I_n}{r_n}\eta$ is derived in terms of X and then Equation (103) is used for the value of X, reducing to the common denominator, D_n, of Equation (104). As before, the factor, $(1 - \cos \phi_1)$, can be taken out and $\left(2 \sin \frac{\phi_1}{r_n^2}\right)$ inserted in the bracketed term; then by use of the formulas, $\sin 2\phi_1 = 2 \sin \phi_1 \cos \phi_1$ and $2 \sin^2 \phi_1 = 1 - \cos 2\phi_1$, a material reduction can be effected, giving,

$$\frac{E I_n}{r_n}\eta = \frac{p_e r_e k^2 (1 - \cos \phi_1)}{D_n}\left[(\phi_1 - \sin \phi_1)\left(1 + \frac{k^2}{r_n^2}\right) + 2.88 \frac{r}{r_n}\frac{k^2}{r_n^2}(\phi_1 + \sin \phi_1)\right]$$

* *Transactions*, Am. Soc. C. E., Vol. LXXV (1912), pp. 937–939.

† *Loc. cit.*, Vol. LXXXV (1922), pp. 240 and 270.

On multiplying by $\frac{r_n}{E\,I_n} = \frac{r_n}{E\,t\,k^2}$,

$$\eta = \frac{p_e r_e r_n}{E\,t} \cdot \frac{(1-\cos\phi_1)\left[(\phi_1 - \sin\phi_1)\left(1+\frac{k^2}{r_n^2}\right) + 2.88\,\frac{r}{r_n}\,\frac{k^2}{r_n^2}\,(\phi_1 + \sin 2\,\phi_1)\right]}{D_n} \quad \ldots (109)$$

or,

$$\eta = c_n \frac{p_e\,r_e}{E\,t}\,r_n$$

(in which, c_n is the value of the last fraction in Equation (109)).

$$D_n = (\phi_1 + \frac{1}{2}\sin 2\,\phi_1)\left(1+\frac{k^2}{r_n^2}\right) - \frac{1-\cos 2\,\phi_1}{\phi_1}$$
$$+ 2.88\,\frac{r}{r_n}\,\frac{k^2}{r_n^2}\left(\phi_1 - \frac{1}{2}\sin 2\,\phi_1\right)$$

$$r_n = \frac{t}{\log_e\left(\frac{r_e}{r_i}\right)};$$

and,

$$\frac{k^2}{r_n^2} = \frac{t^2}{12\,r_n^2} = \frac{1}{12}\left(\log_e\frac{r}{r_i}\right)^2$$

For small values of ϕ_1, seven or eight-place logarithmic tables must be used for determining the first two terms of D_n.

For comparatively thin arches, on putting $r_n = r$ the formula reduces to that previously derived.* In the formula for η, both p_e and E must be expressed in the same unit, pound per square foot, or pound per square inch, in which case η is expressed in feet, since $\frac{p_e}{E}$ is simply a ratio.

The influence of shear seems to be very large for thick arches. Thus, for $t = 30$, $r = 100$, and $\phi_1 = 20°$, Equation (109) gives, with shear included, $E\,\eta = 0.751\,p_e\,r_e$. Omitting the terms containing 2.88 as a factor (that is, the influence of shear) $E\,\eta = 0.266\,p_e\,r_e$. Therefore the deflection including shear is 2.82 times the deflection when the influence of shear is omitted.

The author's Equation (45)† can be reduced to Equation (109), when $m = \infty$ and ε' is replaced by $-\frac{p_e\,r_e}{E\,t}$. When m is finite and σ_x becomes a factor, the author's formula gives a ready solution.

Thrusts and Moments Due to Temperature Changes.—Let e equal the expansion per foot for a rise of 1° Fahrenheit; and t_0 equal the number of degrees of change in temperature.

For a free cylinder, the span of the neutral line will be increased, ($e\,t_0\,2\,r_n \sin\phi_1$) ft. Now regard the arch fixed at the right end and free to move

* *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 270.
† See p. 222.

at the left end, where H, the horizontal force, is just sufficient to bring this end back to its original position. Fig. 31 can be used to illustrate the conditions if $p_e = 0$, $V = 0$ and $P_0 = H$, so that the half-arch is in equilibrium under the forces, H, at A and C, and the couples of moments, M_1 and M_0, at A and C, are just sufficient to prevent rotation of the sections. If L is the elastic work for the entire arch (twice the value given by Equation (98)), the horizontal displacement of the left end toward the right, is,

$$\frac{\delta L}{\delta H} = e\, t_0\, 2\, r_n \sin \phi_1$$

Also, since there is no rotation of the crown section, $\frac{\delta L}{\delta M_0} = 0$. Then,

$$M = M_0 - H r_n (1 - \cos \phi); \quad P = H \cos \phi; \quad S = -H \sin \phi$$

$$\frac{\delta M}{\delta M_0} = 1; \quad \frac{\delta M}{\delta H} = r_n (\cos \phi - 1); \quad \frac{\delta P}{\delta M_0} = 0;$$

$$\frac{\delta P}{\delta H} = \cos \phi; \quad \frac{\delta S}{\delta M_0} = 0; \quad \frac{\delta S}{\delta H} = -\sin \phi.$$

Differentiating under the integral sign as before, there is derived from the condition, $\frac{\delta L}{\delta M_0} = 0$, the relation,

$$M_0 = H r_n \left(1 - \frac{\sin \phi_1}{\phi_1}\right)$$

and from the condition, $\frac{\delta L}{\delta H} = 2\, e\, t_0\, r_n \sin \phi_1$, after reduction, the relations,

$$H = \frac{2 \sin \phi_1}{D_n} \left(E e t_0 \frac{k^2}{r_n^2} t\right);$$

and,

$$D_n = \left(\phi_1 + \frac{1}{2} \sin 2\, \phi_1\right) \left(1 + \frac{k^2}{r_n^2}\right) - \frac{1 - \cos 2\, \phi_1}{\phi_1} + 2.88 \frac{r}{r_n} \frac{k^2}{r_n^2} \left(\phi_1 - \frac{1}{2} \sin 2\, \phi_1\right)$$

These formulas for H and M_0 are for a free arch with a rise of temperature of t_0 degrees Fahrenheit above an assumed mean, the arch not being restrained by the cantilever action. The latter will be considered subsequently. Of course, for a fall of t_0 degrees of temperature, t_0 is negative and the direction of H is reversed.

Radial Deflection at Crown Under a Rise of Temperature of t_0 Degrees.— For a free cylinder the span of the neutral line will be increased $2\, e\, t_0\, r_n \sin \phi$, and its rise, $h = r_n (1 - \cos \phi_1)$, will be increased $e\, t_0\, r_n (1 - \cos \phi_1) = h\, e\, t_0$. In this stage no work is done. Now, for this expanded arch, regard the right end as fixed and the left end as free and as subjected to the forces, H and M_1, just derived, which bring the section back to its original position. By the "work" method, the corresponding value of the crown deflection, η', is found.

As deflection toward the center is positive, the total radial deflection of the crown is,

$$\eta\, t = \eta' - h\, e\, t_0$$

To find η', since,

$$P = H\cos\phi, \text{ and } S = -H\sin\phi$$

$$M = M_0 - H r_n (1 - \cos\phi) = H r_n \left(\cos\phi - \frac{\sin\phi_1}{\phi_1}\right)$$

The solution proceeds now exactly as in the writer's previous paper* and leads to the formula,

$$\eta' = \frac{2\sin\phi_1}{D_n} h\, e\, t_0 \left[\frac{1}{2}(1+\cos\phi_1)\left\{1 + \frac{k^2}{r_n^2}\left(1 - 2.88\frac{r}{r_n}\right)\right\} - \frac{\sin\phi_1}{\phi_1}\right]$$

Therefore, the full deflection reduces to,

$$\eta_t = \eta' - h\, e\, t_0 = -c_n (r_n\, e\, t_0)$$

in which, c_n is the coefficient of $\frac{p_e r_e r_n}{E t}$ in Equation (109).†

Hitherto, the horizontal arch fixed at the ends, under the change of temperature has been considered as free, or unrestrained by the dam acting as a cantilever. Necessarily, this constraint must be considered. When this is done, the problem offers difficulties. The writer has given‡ a solution after the hypothesis of B. A. Smith,§ M. Am. Soc. C. E., in which the dam with the up-stream face vertical, under a rise of temperature of t_0 degrees, was assumed (1) to have a "cylinder expansion" on a smooth base, with a radial up-stream displacement of the neutral line equal to $(-r_n\, e\, t_0)$; and then (2) if p_e' is the unit normal force exerted at the extrados by the dam acting as a cantilever, assumed constant for any one horizontal arch, the down-stream displacement

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* *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), pp. 240 and 270.

† This result can be obtained in another way. When a cylinder is subjected to the rise of temperature, the unit elongation along or parallel to its medial line is $e\,t_0$. Assume that the expansion is due to a tangential tension, $t.Eet_0$, on each cross-section and that the arch is brought back to its original position at the left abutment by the forces, H and M, acting there. The tangential component at (r_n, ϕ), the moment, and the shear are:

$$P_t = H\cos\phi - t\,E\,e\,t_0 = \frac{2\sin\phi_1}{D_n} E\,e\,t_0 \frac{k^2 t}{r_n^2}\cos\phi - t\,E\,e\,t_0$$

$$M_t = \frac{2\sin\phi_1}{D_n} E\,e\,t_0 \frac{k^2 t}{r_n}\left(\cos\phi - \frac{\sin\phi_1}{\phi_1}\right)$$

and,

$$S_t = -\frac{2\sin\phi_1}{D_n} E\,e\,t_0 \frac{k^2 t}{r^2{}_n}\sin\phi$$

For an arch subjected to normal loads, p_e, it has been found that,

$$P = -\left(\frac{2\sin\phi_1}{D_n} p_e r_e \frac{k^2}{r_n^2}\cos\phi - p_e r_e\right)$$

$$M = -\frac{2\sin\phi_1}{D_n} p_e r_e \frac{k^2}{r_n}\left(\cos\phi - \frac{\sin\phi_1}{\phi_1}\right)$$

and,

$$S = \frac{2\sin\phi_1}{D_n} p_e r_e \frac{k^2}{r_n^2}\sin\phi$$

Observe that,

$$\frac{P_t}{P} = \frac{M_t}{M} = \frac{S_t}{S} = -\frac{t\,E\,e\,t_0}{p_e r_e}$$

hence, the crown deflections must be in the same ratio, or

$$\frac{\eta_t}{\eta} = -\frac{E\,t\,(e\,t_0)}{p_e r_e}$$

therefore,

$$\eta_t = -c_n \frac{p_e r_e r_n}{E t} \cdot \frac{E\,t\,(e\,t_0)}{p_e r_e} = -c_n (r_n\, e\, t_0)$$

‡ *Transactions*, Am. Soc. C. E., Vol. LXXXIV (1921), pp. 80–81.

§ "Arched Dams," *Transactions*, Am. Soc. C. E., Vol. LXXXIII (1919–20), p. 2027.

corresponding of the arch, η, is $c_n \frac{p_e r_e r_n}{E t}$; hence, the resulting displacement of the crown section is,

$$r_n \left(- e t_0 + c_n \frac{p_e r_e}{E t} \right)$$

The advantage of assuming a cylinder expansion in Assumption (1), is that it ensures a constant p_e' for any one arch. The disadvantage is that the forces, p_e', cause the same displacement all around, and thus the dam is not vertical at the abutments, as it should be. The solution cannot be regarded as exact.

Another approximate solution will now be given, in which the cylinder expansion is not used in the first stage. In Fig. 35, let $A\ D\ B$ represent the plan of the neutral line of a free arch fixed at the ends, which, under the temperature rise, takes the position, $A\ C\ B$. The up-stream displacement of the free arch, as just shown, is, $\eta_t = D\ C = - c_n\ (r_n\ e\ t)$, and the dam at the abutments remains vertical. However, in considering the action of the dam as a cantilever, its forces, p_e', can no longer be regarded as constant for the same arch. They probably decrease in going from the crown to the abutments. Experiment will have to determine this matter, not only for this case, but for the general solution of an arch under water pressure alone.

To effect a solution in the present state of knowledge, it is necessary to assume p_e' as constant; then,

$$\eta = I\ C = c_n \frac{p_e'\ r_e\ r_n}{E\ t}.$$

Therefore,

$$D\ I = c_n\ r_n \left(- e\ t_0 + \frac{p_e'\ r_e}{e\ t} \right)$$

in which, deflection up stream is negative, and down stream, positive. The value of p_e' for the level considered, being the pull or resistance of the cantilever to the expansion of the arch, is supposed to be known from the solution of the arch.

The movements indicated in this last solution are clearly shown in Fig. 36, representing a vertical radial section at the crown, in which, $F\ D$ represents the original position of a vertical line through D (Fig. 35), which is moved to $E\ L$ under the expansion of the free arch, a distance, $D\ C = F\ E = - c_n r_n e t_0$. The cantilever forces, p_e', acting on an arch at the level, C, force C back a distance, $I\ C = c_n \frac{p_e'\ r_e\ r_n}{E\ t}$, so that $D\ I$ is the actual deflection reckoned from $F\ G$. The elastic curve is $F\ I\ H$.

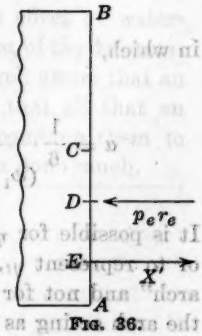

FIG. 36.

If Fig. 36 is revolved 180° about $F\ G$, it can represent the displacements due to a fall of temperature and shrinkage, e' per unit of length, the cantilever

resistance now acting away from the center. The resulting deflection down stream is,

$$D\,I = c_n\,r_n\left[-e\,t_0 + e' - \frac{p_e'\,r_e}{E\,t}\right],$$

in which, t_0 is negative and $-e\,t_0$, positive.

As none of these formulas can be regarded as exact, it is hoped that experiment will come to the rescue and aid in producing a practically exact formula. To consider the case for combined water pressure, shrinkage, and temperature rise, suppose the solution of the arch dam gives, for an arch at a particular level,

p_e = normal pressure on the arch, in pounds per square foot, due to water; and

p_e' = normal pull of the cantilever or its resistance to the temperature expansion.

Then, adding the displacement down stream due to the water to the displacement due to the temperature rise, the total radial displacement at the crown will be:

$$D = c_n\,r_n\left[-e\,t_0 + e' + \frac{(p_e + p_e')\,r_e}{E\,t}\right]$$

This is down stream when positive and up stream when negative. For a fall of temperature, shrinkage, and water pressure, the formula applies on taking t_0 and p_e' as negative. The formulas, of course, are only valid where no cracks are formed, either from the shrinkage or from the exertion of too much tension.

If precise observation at the crown of an arch gives the total deflection, D, the solution of the last equation for $(p_e + p_e')\,r_e$ gives the total thrust. The Smith hypotheses (which may prove the better) lead to the formulas previously given by the writer.* The moment at the crown can be expressed in terms of the deflection, as follows:

$$M_0 = -\,a\,\frac{E\,t^3}{r_n\,h}\,\eta$$

in which,

$$a = \frac{1}{6}\,\frac{\sin\phi_1\left(1 - \dfrac{\sin\phi_1}{\phi_1}\right)}{(\phi_1 - \sin\phi_1)\left(1 + \dfrac{k^2}{r_n^{\,2}}\right) + 2.88\,\dfrac{r}{r_n}\,\dfrac{k^2}{r_n^{\,2}}\,(\phi_1 + \sin 2\,\phi_1)}$$

It is possible for η to represent the deflection due to the water pressure alone, or to represent η_t, the deflection due to a change of temperature for a "free arch" and not for one the expansion (or contraction) of which is resisted by the arch acting as a cantilever. Experiments should be made to determine the coefficients in the formulas referring to temperature changes on the dam with the reservoir empty.

* *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 277.

Then, if D_t = the observed crown deflection,

$$\eta_t = D_t - c_n r_n \frac{p' r_c}{Et}$$

As η_t varies directly with t_0, it can be found for any t_0 different from that of the observation.

Next, with the reservoir full, if possible, observe the deflection, η, due to water pressure only when there is no change of temperature. If this is not feasible, let D be the observed crown deflection for combined water pressure and t_0 degrees change of temperature; then the formula for D can be written in the form,

$$D = \eta_t + \frac{p_c + p_c'}{p_c} \eta$$

whence,

$$\eta = \frac{p_c}{p_c + p_c'} (D - \eta_t)$$

and, finally,

$$M_0 = - a \frac{E_t^3}{r_n h} (\eta_t + \eta)$$

The author's notes on the "Secondary Arch"* are particularly full and useful. The approximate solution of arches of varying section appears to be as brief as the subject permits. Castigliano's method, utilizing Simpson's formula, might be adapted to the case, but possibly the solution would be longer.

The writer is unable to agree with the author that as $2\ \phi_1 \doteq 0$, the intrados at the crown can ever suffer compression, since the arch then reduces to a straight beam, fixed at the ends and loaded uniformly, under which condition the center on the intrados side always experiences tension.

With regard to the conservation of plane sections of a beam, column, or arch, to stress-strain curves, and to the influence of shear in warping plane sections, George F. Swain, Past-President, Am. Soc. C. E., has thrown† a flood of light on a subject not well understood.

The author has written very convincingly of the serious effect of water-soaking in an arch dam. There can be no doubt of its influence, of the deformation of the foundation, and of the variation in E. He seems well aware that an exact solution of the arch dam is not to be looked for, so that all that an engineer can do is to examine the various influences and combine them to effect a practical solution. In that direction Mr. Jakobsen has done much.

* See p. 241.

† "Strength of Materials," pp. 234–242.

THE REINFORCED CONCRETE ARCH IN SEWER CONSTRUCTION: A REVIEW OF PAST PRACTICE IN DESIGN AND AN ACCOUNT OF RECENT STUDIES IN ST. LOUIS, MISSOURI

Discussion*

By Messrs. Charles S. Whitney and Jacob Feld.

Charles S. Whitney,† M. Am. Soc. C. E. (by letter).‡—In making his analysis of the stresses in sewer arches, Mr. Sharp has assumed that the footings of the top arch rest on an elastic foundation and that their rigidity is not great enough to produce a fixed-end condition. He has correctly considered that the true condition is one of partial restraint, and he has assumed limiting conditions for the purpose of determining maximum stresses. The writer believes this to be the proper method of attack, but he has a somewhat different conception of the action of the loads on the arch.

For the majority of elastic structures, when the combined effect of several different loads is to be determined, it is possible to consider the effect of each load separately and then to combine the results. This does not seem to be true of a sewer arch surrounded by a fill. For instance, the stresses produced by filling the sewer with water will not be the same before and after the back-fill has been placed. In the former case the arch acts as a ring restrained only at its supports, whereas in the latter case the back-fill resists the deformations which would be produced in the arch ring by the water pressure.

The effect of the outward pressure of the water is, therefore, partly neutralized by the increase in inward pressure of the fill. In resisting an outward movement, the fill may exert a much greater pressure than would be developed against a fixed surface. The degree to which the water pressure will be balanced will depend, of course, on the relative stiffness of the arch ring and its surrounding envelope of earth. It seems probable that, in the case of a large sewer, the arch ring will be flexible enough so that a considerable part of such water pressure would be transmitted directly through it to the earth. Some idea of the importance of this effect could be obtained by determining the deflection of the arch as a free ring and the compressibility of the earth.

The fill placed at the top of the arch will increase the horizontal pressure from the fill by causing the sides of the arch to bulge outward. This increase in side pressure will assist the arch in carrying the weight on top. The reinforced concrete sewer has one important advantage over the unreinforced sewer in that

* Discussion on the paper by Charles E. Sharp, Jr., Esq., continued from December, 1925, *Proceedings*.

† Cons. Engr., Milwaukee, Wis.

‡ Received by the Secretary, December 18, 1925.

it possesses much greater flexibility, which permits an adjustment of the external earth pressures and a reduction of the bending and shearing stresses. If the fill is deep, the downward deflection of the crown as the top fill is placed would tend to develop some arch action in the fill and relieve the top pressure, while the bulging of the sides would increase the side pressure. It is conceivable, therefore, that in a more or less plastic soil the pressure on the arch would be similar to that of a fluid the weight of which is somewhat less than that of the soil.

Ten years ago, the writer devised a graphical method for the analysis of large concrete conduits* and at that time stated that:

"The consideration of temperature and rib-shortening stresses in a conduit in an earth fill would seem to be a useless refinement when the external forces are of such an indeterminate nature. No experiments have revealed any accurate law by which the pressures in a deep fill effected so much by arch action and varying conditions can be figured."

Although no more information is now available as to earth pressures on conduits, the successful completion of large conduits is an indication that the assumptions made in their design are safe. For that reason, the record of the method of design given in Mr. Sharp's paper is very valuable and indicates a step forward.

JACOB FELD,† ASSOC. M. AM. SOC. C. E. (by letter).‡—The author deserves the thanks of the profession for his attempt to place the analysis of a heretofore empirically designed structure on a scientific basis. As stated in the paper, designers are often so "experienced" with the shape and general bulk of the structure in mind that they do not even attempt to porportion each individual structure from the point of view of efficiency in strength, but rather copy an existing example, with some additions, "on the side of safety", as being strong enough. Especially is this true of those designs which are not usually included in textbooks, or elementary structural courses.

It is important to remember that the sewers for which the design methods in the paper were developed are 32 ft. wide, much larger than many of the small bridge or culvert structures that are sometimes designed even for secondary stresses. It seems incongruous that such sewers do not receive more attention; and the vast length of the smaller sizes that are built makes it only proper that no useless materials be required for their construction.

It is to the benefit of the engineer and the builder, as well as of the general public that structures be designed for safe life but at the same time economically and efficiently. An unbalanced design, evidently too strong in some links for the conditions covering its use, is often employed for a worse set of conditions under which the weak link in the design becomes apparent, with serious consequences.

In designing structures of the class that includes arch sewers, enough mathematical analysis is available to determine with fair accuracy all the

* "Reinforced Concrete Conduit Analysis Simplified by Theory of Displacements," *Engineering Record*, October 16, 1915, p. 486.

† Cons. Engr., New York, N. Y.

‡ Received by the Secretary, December 28, 1925.

internal stresses for any combination of assumed loadings. When the engineer is dealing with true solids, liquids, or gases, he can make assumptions for possible load and reaction forces readily and within narrow limits. This results from the data accumulated in centuries of experimental work. Such limits must be made wider when the materials do not fall into one of the so-called three states of matter. Some of the laws of solids can be borrowed for the design of structures acted upon by soils, with, perhaps, some modifications from the laws of liquids.

The author describes the back-fill as a "plastic soil". He then uses what data are available regarding soils, none of which, unfortunately, covers exactly the type that the ordinary engineer would call a "plastic soil". As a result, the author opens a wedge for criticism, which probably is more severe (although not properly so) than if he had used data obtained from experiments with pure solids or pure liquids. He probably realizes that his analysis is only an approximation, to be made more accurate for any one set of conditions, as such are determined.

There are, however, several items in the paper that might have been slightly altered to make it more uniform in principle. In assuming conditions of support of the arched sewer* "current practice was followed", entirely different conditions being accepted for compressible soils and for rocks. This is not necessary; rock and soils as bearings differ only in degree. The same method of attack can be made to cover both classes, taking into consideration the properties (quantitatively) of the supporting material. The choice of a certain type of support, spread footing, abutment, or what not, does not bring into actual existence the basis for such assumptions. It is not apparent from the general names given to the soils why there was necessity for such varied types of footings; a more careful classification, with whatever experimental data on the physical properties of the soils encountered may be available, would be helpful to a full understanding of the paper.

The assumptions of loading are certainly not justifiable. The vertical load on a buried pipe depends on a great number of factors. It seems that the author hoped to take an average condition by assuming that the full weight of the superimposed earth bears on the sewer arch. This may not be the maximum condition. In loose granular soils, the vertical load on the pipe, if it be placed on a non-solid sub-grade, may be far greater than the author's assumption; and the maximum vertical loads may act simultaneously with either a maximum horizontal load or even with no horizontal load, depending on the state of affairs along the sides of the arch.

It does not seem logical to argue that the usual formula for soil pressure on a retaining wall, as checked by the writer's experiments at the University of Cincinnati, apply to this case. The pressure on a retaining wall cannot act unless there is an unbalanced displacement in one direction (normal to the plane of the back of the wall). This displacement, to agree with the conditions under which the theory was developed and under which the experiments were performed, must occur on the full height of the wall, although not necessarily equal in magnitude over the entire height.

* *Proceedings*, Am. Soc. C. E., August, 1925, Papers and Discussions, p. 1085.

In the case under consideration these conditions do not apply. Above the arch there is merely soil acting against soil; the influence of the discontinuity in the soil body, at the sewer arch, is a much more complex phenomenon and not even approximated by the ordinary earth pressure formula for retaining walls. It seems that the formulas should give for the horizontal pressures values that are too high for this case. This does not correspond to a design on the "side of safety".

That the assumed loadings are not considered sufficient to cover maximum conditions is the practical admission made by the choice of low unit stresses. Concrete stress at 540 lb. per sq. in., in compression under flexure, only 60% of the value used in some City Codes, and far below the recommendations of the Joint Committee on Standard Specifications for Concrete and Reinforced Concrete, seems to be far too low to balance the rigid stress analysis.

It seems to the writer that a far more certain design could be made by trying two sets of assumed loadings, the maximum and minimum possible, as determined from what experimental data are obtainable, especially from the tests on buried pipes and culverts, and averaging these two designs, not arithmetically, but in accordance with actual materials and conditions found at the point in question.

The writer feels that, after designing a sewer arch by the method and with the loading and restraint conditions assumed in the paper, there is no certainty that even an approximation to the ideal efficient design of uniform strength has been reached. The author is in no way to be blamed for this lack of confidence; it is caused solely by the insufficiency of available information. When a project of the size described is planned and undertaken, it certainly seems that some contribution to the necessary fund of experimental data on a practical and large scale should have been made. It would have cost only a small fraction of the total amount spent and probably have resulted in a monetary saving on this one project alone of many times the cost of the experiments.

THE ST. LAWRENCE WATERWAY TO THE SEA

Discussion*

By E. G. Walker, M. Am. Soc. C. E.

E. G. Walker,† M. Am. Soc. C. E. (by letter).‡—The writer has read this paper with interest, for there are few individual engineering proposals in the limelight at present which have so general an interest, or which, if carried out, would have such far-reaching consequences. To appraise fully the magnitude and nature of the proposals for regulating the Great Lakes and the St. Lawrence River requires complete inspection and first-hand acquaintance with all the sites referred to by the author. The writer is at a disadvantage in this discussion in not possessing such acquaintance; hence, he prefers to confine his statements to more general matters.

Critics of plans for the greater utilization of the St. Lawrence question not so much the possibility of carrying out the works as the likelihood of their full use when completed. Criticism of details is seldom or never heard. No doubt this is due in part to the fact that none of the works proposed is of a startling nature or exceeds in general form and scope the very fine results which have already been achieved both in regard to power development and as to the provision of navigation facilities. At the same time the construction problems involved are not simple. The condition which makes the St. Lawrence unique as a power-producing river is the enormous magnitude of the reservoirs out of which it flows. While this gives it a minimum fluctuation of flow, it also means that there are no slack-water periods, such as occur on most rivers, of which advantage can be taken to carry out under-water works, and the whole flood volume has to be provided for in any plan for controlling the water during construction. It is not suggested that these and similar difficulties cannot be overcome. The writer merely raises the point because it is one which is apt to be overlooked. On the other hand he considers that a very great reason why the whole proposition should be given the fullest and most rational consideration is because it is one that does not involve any new or startling departure, but only requires the construction of works similar in character to those which have been carried out already in various parts of the world. The total expenditure proposed is vast and obviously cannot be undertaken without a very full study by men of high qualifications; but should the work be decided on, each unit into which it is divided is a solvable engineering problem. Although, at present, the total cost is merely a guess, it is certain that the proposals can be carried out in detail, even if some of them may prove to be difficult.

* Discussion on the paper by Francis C. Shenehon, M. Am. Soc. C. E., continued from January, 1926, *Proceedings*.

† London, England.

‡ Received by the Secretary, December 15, 1925.

The main object of the paper being to advocate the construction of a deep-water navigation from the head of Lake Superior to the sea, it is disappointing to find inadequate consideration paid to the primary features of such a waterway. The paper leaves an impression that the premises on which any plan for a waterway must be built have been considered far too lightly. The idea of being able to take a large ocean-going ship into the heart of a continent is one which inevitably must make a strong appeal even to the dullest imagination, and the glamour of the idea is apt to spread in an unwarranted manner and to mask opposing features which otherwise would stand out in a clearer light. Enthusiasm for the greatness of the development and for its possible widespread influence seems in the present instance to have drawn attention away from material facts which are of much too great importance to be passed over lightly.

As evidence of this the writer notes how little idea is given by the author of the type of ship which it is proposed to use on the waterway. He states*:

"It will be assumed that engineers will solve the problem of composite vessels to traverse the fresh-water seas and the salt seas; that the evolution of this vessel will come in the ten years between the definite decision to construct the waterway and its completion; and if not in this decade then in some of the early decades of the century."

Instead of making the assumption that it would be possible to find a type which can be used on the waterway when built, why not give some consideration to the matter now? If the search for a type of ship suitable to the special conditions of the case and for methods of handling such ships and their cargoes is successful, then it is at least known that if the waterway is provided it can be used.

From the discussions on this point the writer concludes that the familiarity of the people near the Great Lakes with the craft they use has distorted the first of these problems into one of needless complexity. The Great Lakes freighter is a unique type of vessel which has been evolved to suit the special conditions obtaining there. It differs from the ocean-going steamer as much as a street car differs from a railroad coach and for a similar reason. The Lakes freighter is free from many of the conditions imposed by ocean service. The result is that it has been possible to evolve a vessel of comparatively light dead weight and of high efficiency both in regard to stowage of cargo and to handling it. With such ships the local traffic of the Great Lakes is transported in the highly economical manner that has justified the installation of expensive special machinery for loading and discharging. It is agreed, and rightly, that these ships are not suitable for general ocean travel, but by some inexplicable inversion of reasoning, it is then assumed that the opposite proposition is also true, namely, that ocean steamers would not be suitable vessels for going up the St. Lawrence Waterway to ports of the Great Lakes.

To any engineer who has been concerned with ships or with the provision of harbor facilities for sea-borne commerce, this idea must appear untenable. Admittedly the ocean-going ship could not compare with the Lakes freighter for interlake services. It would not be used for such work; it would deal with

* *Proceedings*, Am. Soc. C. E., September, 1925, Papers and Discussions, p. 1268.

cargoes which the Lakes ships could not handle; but both types of vessel use similar prime movers, propel themselves by screws of similar kind, and are steered and maneuvered similarly. The ship which can cross the high seas can equally well traverse the Great Lakes and negotiate any connecting waterways, provided the dimensions of these are suitable. One is at a loss, therefore, to understand whence arises this fetish of the hybrid ship. If cargo is to be consigned overseas it must be sent in an ocean-going ship; if inland produce is not to be sent directly overseas there is no reason for constructing a deep-water navigation to the farthest possible point inland.

The only excuse for attempting to bring the hybrid into being is that it would aim at combining the sea-going qualities of the one type with the efficiency of stowage and ease of loading of the other. Such an aim is largely a myth. The basis of design of all cargo ships is a complex compromise between a number of factors, often mutually antagonistic, such as seaworthiness, speed, ease of handling, cargo capacity, kinds of cargo to be transported, radius of action, etc. and in arranging the details account has to be taken of extreme conditions. In the case of ocean-going ships designed for the projected St. Lawrence navigation such conditions are provided by the high seas in practically every detail, except over-all dimensions, even to the relatively minor condenser problem referred to in the discussion. The principal possible complicating condition which a waterway would cause would be tonnage measurement rules. These, of course, might materially influence the design of the ships, but can hardly be discussed at the present stage. There is no reason to anticipate that if the proposed waterway should be constructed it would be found necessary to introduce freak tonnage regulations. The inevitable conclusion, therefore, is that if the St. Lawrence Waterway should be opened as a work of the magnitude contemplated by the author, it will be used, as have all waterways of a similar kind thus far constructed, by the normal type of ocean steamers which would voyage between overseas ports and the various ports along the waterway.

Whether the present terminal facilities in the Great Lakes ports are suitable for the contemplated traffic is a matter which has not been mentioned in the paper. The ocean steamer differs considerably from the Lakes vessel in the arrangement and dimensions of hatchways and the facilities available for stowing and discharging cargo, and must continue so to differ because of the provisions which have to be made to meet open sea conditions. Furthermore, the opening of the Great Lakes ports to overseas commerce would probably have the effect of causing a large increase in the amount of general cargoes passing through these ports in comparison with the special cargoes which are now the principal ones handled. Advocates of the proposed waterway do not seem to have considered adequately that one of the main effects of a deep-water direct connection to the sea might be expected to be the opening of numerous new inland ports to the exportation and importation of sea-borne goods, and that if this movement were successful, it would alter very greatly the average character of the shipping using the Great Lakes ports. This would mean that terminal and port facilities suited to overseas traffic would

have to be provided at lake-side ports, and in many cases might entail considerable expenditure for re-arrangement of wharves, warehouses, transit sheds, cranes, sidings, etc., and for the provision of new works, in order to handle products which at present are not carried by water at all as far as the territory of the Lakes is concerned.

Assuming the proposed waterway to be constructed and provision made for accommodating any type of ship, it is unlikely that vessels would enter and incur the expenses of lockage and other dues (which, in view of the great total lift of 500 to 600 ft., must be heavy) unless they were on definite charters. Statements that have been made as to the possibility of tramp steamers in search of cargo visiting Great Lakes ports in the same way as at ocean ports are far-fetched and unlikely. The costs of operating such a navigation as that outlined in the paper must be fairly high. They must be sufficient to increase very materially the cost of transport over the 1 340 miles above Montreal.

The author has taken for the purposes of this paper a depth of 25 ft. with provision for a future increase to 30 ft. and in so doing has used correspondingly moderate dimensions for the lengths and breadths of locks and other permanent works. Apparently, his basis is a Lakes freighter 600 ft. long, with a 60-ft. beam, 24-ft. depth, and a cargo capacity of 10 000 tons, although as already pointed out it is unlikely that such a vessel would be used for overseas commerce. It is probable that provision for 10 000-ton ships would be more than sufficient to meet the requirements for a long time. Of the world's ships 50% do not exceed 1 000 tons and although there is a tendency to increase the size of vessels of between 5 000 and 10 000 tons it is unlikely there will be, at any rate for as long a period as the present generation need consider, more than a small minority which could not use a waterway large enough to accommodate a 10 000-ton, ocean-going ship. To make certain of this, however, it might be well to give increase of breadth to all permanent works, since the vessel selected by the author for his standard is narrow in proportion to its length. A 10 000-ton intermediate ship would normally have a beam of about 60 to 70 ft. and be appreciably shorter than the Lakes steamer. The dimensions proposed for locks, namely, 800 by 80 by 30 ft., should be ample.

In discussions on the power development aspect of the St. Lawrence Waterway, objection to the plans has been taken on the score that so large an output could not be used. Accepting the estimate given in the paper, the author appears to conclude that an ultimate total of nearly 7 000 000 h.p. is available, made up as follows:

Present development:	
Canadian	700 000 h.p.
American	570 000 h.p.
Total	1 270 000 h.p.
Possible additional power (with regulation)	4 424 000 h.p.
Possible improvements at Niagara	1 160 000 h.p.
Total	6 854 000 h.p.

The writer has no figures at hand relating to American power developments, but taking the current figures for the Canadian Provinces of Ontario and Quebec only, the following water power is already in use:*

Ontario	1 445 000 h.p.
Quebec	1 116 000 h.p.
Total	2 561 000 h.p.

This power serves a population of about 5 500 000 and includes the 700 000 h.p. given in the author's figures. It is equivalent to about 0.47 h.p. per person. On this basis there would remain 6 154 000 h.p. to be utilized, requiring a population of more than 13 000 000. The power requirement in the Provinces of Ontario and Quebec is greater than the average for the United States and Canada, which works out as 0.44 and 0.35 h.p. per head, respectively. It is probably greater than the average for the whole territory tributary to the Great Lakes and the St. Lawrence. Hence the development of the two countries would have to be greater even than is implied by the indication of this simple statistical calculation. Bearing in mind that this population figure is of the magnitude of the total increase of the population of the United States in the last ten years, one gets some impression of the size of the problem involved in harnessing the whole of the St. Lawrence power.

There are two potential competitors with the present plan that have to be reckoned with, namely, (1) other hydro-electric power; and (2) power from coal. Roughly speaking, these resolve themselves into Canadian and American competition, respectively. The large undeveloped sources of power in Canada and particularly in the Province of Quebec are well known and the 5 000 years' supply of bituminous coal in the United States to which the author refers may well be formidable competitors. Remarkable advances have been made in recent years in the economical use of coal as a source of power and it is noteworthy that some of the most economical power stations in the world are situated in the area treated in the paper. The writer does not agree, therefore, with the reasoning that because supplies of coal are ample, no concern need be felt about saving coal. No hydro-electric power plan can oust steam power unless it can make an appreciable reduction in the cost of supply of current. The fact that such large deposits of coal are available seems to the writer to make the use of that material as a source of power a big factor in considering the development of the huge hydro-electric propositions within a relatively short distance of the largest coal districts of the country. One would like to have much closer estimates of costs of power than are given in the paper, and also to have for comparison figures as to costs of generating from coal, before venturing on definite opinions.

In the Province of Quebec the power supply situation is different, but none the less subject to conditions which must be very carefully considered in connection with this St. Lawrence project. The Province of Quebec is only at the beginning of its development, and it is hardly possible to forecast exactly along what lines such development will take place; but the northern parts,

* Annual Report, Dominion Water Power and Reclamation Service, 1923–24.

which are yet almost unexplored, contain much mineral wealth that must inevitably be developed in the course of time and there are large timber and agricultural resources likewise awaiting development. Further, the Province contains sources of hydro-electric energy that are certainly much more than sufficient for its present needs, and that may well afford all the power required for mining and other developments for a long time to come. In the majority of cases the potential industrial developments are placed nearer the undeveloped power resources of Northern Quebec than to the sites referred to in the paper. It is certainly more likely that the power required for opening up what must develop some day into one of the most important metalliferous areas on the whole Continent will be supplied from the sources near at hand before having recourse to the more remote potential supply from the upper reaches of the St. Lawrence.

Most engineers will regret that the utilization of this wonderful water system cannot be treated as a purely economic and technical problem. It is unfortunate that politics, international as well as national, are so dominantly involved, for the considerations thus brought into the subject are not those best calculated to advance the interest of humanity. Be that as it may, however, the situation remains. The writer is not concerned here with political matters except so far as their influence on the large engineering proposition is concerned. From his examination he has reached a few simple conclusions:

1.—As the greater part of the channel (including most of the proposed power development) of the St. Lawrence is entirely within the Province of Quebec, the attitude of that Province must be inevitably a dominating factor in the scheme.

2.—Quebec opinion generally is that the Quebec Port of Montreal would lose much of its trade if shipping were free to pass on up to the head of the Great Lakes.

3.—The people of Quebec realize the potentialities of their Province and also the overwhelming importance, living as they do in a cold country with no coal deposits, of controlling fully all the sources of power they have, and they are not likely to agree to any proposal for generating power for export.

4.—With the passing of time the industrial activities of Quebec will extend more and more toward the north and northeast, that is, in general, down the St. Lawrence away from the part of the river available for power generation.

That the opening of a waterway for large ships up to Lake Superior would injure the Port of Montreal, the writer does not believe. To extend navigation above Montreal would doubtless divert a certain amount of traffic from that port to points above it; but, on the other hand, the waterway may quite well be expected to compensate by creating traffic not now existing and to bring into being new lines of commerce, from which Montreal will benefit in common with all the other ports along the waterway.

On the question of power development, however, the writer does not believe that at present progress will be served best by the St. Lawrence plan. Doubtless it will come, for ultimately the world will not be able to afford to let any

possible source of energy run to waste; but before this time, much more development will have occurred in the northern parts of the St. Lawrence Basin, and the sources of power in these areas may well be utilized before economic necessity brings about the harnessing of the St. Lawrence. Development is gradually pushing seaward along the north bank of that river, and power plants exist on a number of its northern tributaries. The writer believes that there is greater probability that this development will continue than that the St. Lawrence power plan as outlined in the paper will be undertaken in the near future.

He does not lose sight of the fact that on the south side of the St. Lawrence, in a direction east from Montreal, there is an area of constantly increasing industrial development. Here, however, the proximity of Montreal on the one hand and of industrial areas over the American border on the other has been the cause of growth rather than the exploitation of natural resources that must ultimately direct the trend of progress in Northern Quebec. If, as seems likely, export of power is prohibited by the Government of Quebec, it is open to question whether the territory south of the St. Lawrence will be able, for many years to come, to utilize more than a small fraction of the power developed under the author's proposals.

Regulation of the Great Lakes in the interests of lake navigation is obviously a matter that stands on quite a different footing. There is everything to be said in favor of proposals that seek to maintain the water levels of the various lakes to relatively constant heights; but this is a matter that can be considered apart from the waterway proposal or the development of power on the St. Lawrence.

HIGH SPECIFIC SPEED HYDRAULIC TURBINES IN THEIR BEARING ON THE PROPORTIONING OF THE NUMBER OF UNITS IN LOW-HEAD HYDRO-ELECTRIC PLANTS

A SYMPOSIUM

Discussion*

By Forrest Nagler, M. Am. Soc. C. E.†

Forrest Nagler,‡ M. Am. Soc. C. E. (by letter).§—In reviewing the papers and discussions, more or less confusion in the understanding of the subject of high-speed runners seems to be attributable to an assumption that the various names applied to this type are synonymous. The term "high-speed turbines" or "high specific speed turbines" is not definitely descriptive, nor does the term "propeller type", "axial flow type", "suction type", or any of the individual names attached to the type definitely fix a place in the specific speed scale. For example, high specific speed turbines may have a specific speed of 10 for impulse turbines, 100 for Francis turbines, or 200 for suction turbines. Similarly, a Moody runner may aparently have a specific speed of 90 or 190. A propeller type turbine or an axial flow turbine may apparently have a similar range of specific speed. The suction type turbine is likewise open to criticism, but it is the most definite term, because it describes a certain characteristic which enters not at all, or to a negligible extent, in conventional forms of turbines; and, furthermore, the name signifies a characteristic which gives an indication of its specific speed and design limits, the latter being based on barometric limits.

The writer has no solution to offer for the confusion arising from the broadness of the terms used to identify the various runners. It is desired, however, to point out that a propeller turbine does not necessarily indicate a high specific speed turbine, nor does a suction turbine necessarily cover a type having a specific speed of less than about 140. Naturally, the design features of any one of these types merge gradually into those of another type, without a definite line of demarcation. The foregoing discussion is offered because of the very great need of definitions to permit avoidance of generalizations that are technically incorrect.

The various papers,‖ particularly those by Messrs. Orrok, Moody, Egbert, and Jessop, should go far toward improving the understanding of American

* Continued from January, 1926, *Proceedings*.
† Author's closure.
‡ Hydr. Engr., Allis Chalmers Mfg. Co., Milwaukee, Wis.
§ Received by the Secretary, December 12, 1925.
‖ *Proceedings*, Am. Soc. C. E., August, 1925, Papers and Discussions, pp. 999–1064.

engineers as to the new runner types, their range of characteristics and their limits of application. Mr. Moody's paper* contributes one valuable tool much needed by all those concerned with the construction or operation of water power plants, namely, his simplification of formulas for the variation of turbine efficiency with the variation of size. Previous comparisons, if at all practical, have been largely empirical and the need of a simplification of the intricate and clumsy algebraic expression of the laws has been very great. It would be well if Mr. Moody's expression for the variation of efficiency with the size of units could be embodied in present test codes, not so much by reason of its peculiar application to test code uses, but because that is probably the most practical place for its inclusion as a reference of interest. It has a certain utility in testing, on account of the growing tendency toward careful comparison between model tests and power plant tests.

The runners which Mr. Moody shows are very interesting. The newness of the art of building this type of wheel is indicated by the fact that Mr. Moody's experience is apparently not in accord with that of the writer, who, for example, has not been able to secure as high an efficiency at the higher specific speeds, that is, of more than 150, with the large projected blade area shown in Mr. Moody's design. Conversely, all the large projected-area runners have been very definitely improved in efficiency as their areas were reduced. Not only is this tendency logical, from considerations of wetted surface, the reduction of which has been largely responsible for the development of specific speeds in excess of 125, but the fact is checked absolutely by experience with marine propellers, where the initial perfection of steamship drive resulted from reduction of the blade area. Similar tendencies are found in the development of windmills, sails, electric fans, and blowers. Its parallel is lacking in the case of the steam turbine only, because the primary consideration of steam turbine design is that of keeping the speed low, rather than high, and because friction loss is to a greater or less extent regained thermally.

Mr. Hamilton† is agreeably persistent, but the main basis of difference in opinion is found in the writer's previous comments as to definitions of types. Had there been any positive knowledge of how to secure high specific speed or had specific speeds even remotely approaching 100 ever actually been obtained, there seems to be hardly a reasonable doubt but that the principles involved would have entered into commercial practice, at least to some extent, without waiting 40 or 50 years. Mr. Hamilton's revision of the specific speed curve after 1861, as given in Fig. 66,‡ is not in accordance with established practice and even the inclusion of a single point approaching a specific speed of 100 between 1860 and 1870 would still effect the curve but slightly, because that point did not enter into subsequent practice.

The writer does not take exception to Mr. Hamilton's statement that Mr. Truax knew what he had and knew how he got it. Exception is taken only to the assumption that he had any particularly high specific speed. The original statement by Messrs. Hamilton and Safford in their paper on "The Amer-

* *Proceedings*, Am. Soc. C. E., August, 1925, Papers and Discussions, p. 1009.
† *Loc. cit.*, October, 1925, Papers and Discussions, p. 1677.
‡ *Loc. cit.*, p. 1685.

ican Mixed-Flow Turbine and Its Setting",* credited Truax with a specific speed of 125. The writer's tests showed less than 50 and Mr. Hamilton's tests only 50 or 60 at the point of best efficiency, with this point so low as to result almost inevitably in the wheel being discarded. The Barber wheel could be set so that, with the proper draft-tube, it might show specific speeds of more than 100. It is doubtful whether either the Truax or Austin could ever do so. Their principal lack is in their adherence to the helical blade form as contrasted to the curved form that is so essential to producing the suction principle on which high specific speed depends. This deficiency is exactly comparable in an aeroplane to the sacrifice of, roughly, one-half its lifting power which would result from using a flat wing in place of a curved or ærofoil section. The quoted statement of aviation engineers to the effect that, with sufficient power, one could fly a plank, has no particular counterpart in runner design, however, and nothing that can be applied to the Truax runner as built can bring it into the class which Mr. Hamilton wishes to place it.

The writer is rather surprised at Mr. Hamilton's statement† that "until the latter part of the Nineteenth Century the efficiency of a water wheel was of comparatively little importance". Francis' efficiency of 80% in 1847, and Swain's, of 80% in 1869, according to Mr. Hamilton's diagram, Fig. 65,‡ as well as Risdon's, who certainly was not asleep prior to 1873, offer interesting comments on this point.

Mr. Hamilton criticizes the writer for using a specific speed of 150 for his original high-speed runner, stating that it should be 129, rather than 150. Referring to the original paper entitled "A New Type of Hydraulic-Turbine Runner,"§ it will be found that two runners are given, one of which had a specific speed of 154 and the other of 125, as tested at Holyoke several years previously. The writer has never used anything but the point of best efficiency in comparing runners. If Mr. Hamilton wishes to make tests on the writer's high-speed runner as he has done on the Truax, it is a question of comparing a speed in excess of 200 with one below 100, these being the relative maximum points obtainable. The fact of "W. W. T.", according to Mr. Hamilton,‖ stating that if a diffusing draft-tube should be added with the Truax wheels there would be a gain, does not justify Mr. Hamilton in so adding a draft-tube and crediting it back to Truax as of 1861. The writer has found no instance of any specific speed ever having been obtained in excess of 60 at the point of best efficiency, or 90 at the maximum possible obtainable point, as far back as Mr. Hamilton outlines. If he wishes to include a point as high as 90, as shown in Fig. 66, the basic curve should likewise be raised to cover the maximum obtainable specific speed, disregarding efficiency entirely for the various runners which made up the basic curve.

In Fig. 67,¶ Mr. Barrows contributes a very interesting comparison of wheels. He correctly analyzes Curve *A*, in that it is based on an uncompa-

* *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), p. 1237.
† *Proceedings*, Am. Soc. C. E., October, 1925, Papers and Discussions, p. 1685.
‡ *Loc. cit.*, p. 1684.
§ *Transactions*, Am. Soc. Mech. Engrs., Vol. 41 (1919), p. 829.
‖ *Proceedings*, Am. Soc. C. E., October, 1925, Papers and Discussions, p. 1683.
¶ *Loc. cit.*, p. 1686.

rable method of measurement. As a matter of fact Curve *A* could be readily moved 30 or 40% to the right, simply by taking the runners on which it was based and cutting back their inlet. Similarly, the high-speed runner curve could be moved 100% to the right by taking a mean radius in place of the outside diameter. As a matter of fact Curves *C* and *B* are truly comparative. The writer does not quite agree with Mr. Barrows that the difference in the position of the curves, which means difference in power coefficient, is attributable to a difference in efficiency. The two turbines may be forms of orifices, but they may differ widely in their coefficients. Centrifugal force opposes the flow in the one and does not oppose it in the other. This gives one of the most positive and inherent obstacles to obtaining a large power coefficient with a Francis type with its more or less radial inflow.

This consideration of power coefficient brings up a very interesting problem —one which the writer has frequently presented to students on this subject. In its simplest elements it may be stated thus: What is the maximum theoretical power that can be developed from an orifice 1 in. in diameter under 1-ft. head?

Mr. Jessop's paper* should serve to owners of power plants, as an excellent example of what can be accomplished by proper operation. Probably one of the greatest wastes to-day is due to a lack of appreciation as to what it means to regulate units over a wide range of gate-opening. The curves shown, particularly those of Figs. 36 to 41,† illustrate this operating feature splendidly, although the writer would point out that it is not fair to compare a Francis turbine of 75 specific speed with a high-speed turbine of 175 specific speed, as they would not be used for the same plant. The former is adapted to heads above 50 ft. and the latter to heads below 20 ft. If a Francis runner of 110 specific speed is compared with a high-speed runner of 150 specific speed, the contrast would be more appropriate and the difference less exaggerated.

The writer is rather surprised at Mr. Jessop's conclusion that identical units should be operated at varying loads for best total economy. Most certainly this is not the case except when the load is such that the gates of one of them may be closed entirely. The general law seems to be that with any type of identical units having efficiency curves that are convex upward, all units carrying load should be operated at the same load. Unfortunately, Mr. Jessop's system of operation is only too frequently encountered in practice, due to the fact that it is not particularly easy to measure the relatively small gain for moderate shifting of load with the usual type of board instruments.

Mr. Jessop's comparison of European and Holyoke tests is excellent and has long been needed, not so much for the information of American engineers as for answering a more or less prevalent belief among European engineers to the effect that American test methods are not correct. The greatest progress can be obtained only by co-operation and anything that will make for mutual confidence between the two groups of engineers cannot help but be advantageous. Hence, the value of Mr. Jessop's Fig. 44.‡

* *Proceedings*, Am. Soc. C. E., August, 1925, Papers and Discussions, p. 1043.
† *Loc. cit.*, pp. 1044, 1046–1047.
‡ *Loc. cit.*, p. 1049.

MULTIPLE-ARCH DAM AT GEM LAKE ON RUSH CREEK, CALIFORNIA

Discussion*

By Messrs. B. F. Jakobsen, J. D. Galloway, I. Oesterblom, B. C. Collier, and Fred A. Noetzli.

B. F. Jakobsen,† M. Am. Soc. C. E. (by letter).‡—It would be hazardous in the extreme to conclude anything from the behavior of a single structure, and more especially since other similar structures subject to equally severe conditions, have shown no sign of deterioration.§

It would seem to the writer that some experiments on concrete made up of the material used in the dam and on other concrete, by subjecting it to water pressure and freezing, could not have failed to throw some light on the subject; they might have furnished some explanation as to why this structure has deteriorated while other structures have not. The mere statement that a structure has deteriorated, while another has not, is of no great value to the engineer, who needs to know the why and wherefore—the *modus operandi*, so to speak—so that he can guard against a repetition.

The authors state‖ that the gravity section is heavier than "standard profiles" of gravity dams. There are gravity dams in existence with base widths of from 64% to in excess of 90% of the height. What is meant by "standard profile"? A dam with a base width of 64% has no provision for uplift; but that can hardly be termed standard. In Italy and Switzerland, where there are Government provisions governing the construction and design of gravity dams, uplift must be included in the design and this was also true in France until recently. The new French regulations do not prescribe uplift, if the dam is provided with proper drainage, not only at the foundation, but throughout the structure, since a construction joint may be quite as weak as the joint at the foundation. The new French regulations, however, prescribe such moderate maximum stresses, that for a dam of any magnitude it is the same as if uplift were included. When a base width in excess of 81% is obtained

* Discussion of the paper by Fred O. Dolson and Walter L. Huber, Members, Am. Soc. C. E., continued from January, 1926, *Proceedings*.

† Engr. in Chg. of Dams, Los Angeles County Flood Control Dist., Los Angeles, Calif.

‡ Received by the Secretary, December 14, 1925.

§ See letter by M. M. O'Shaughnessy, M. Am. Soc. C. E., in *Engineering News-Record*, July 30, 1925, p. 194, stating that the Lake Eleanor multiple-arch dam located at an elevation of 4 660 ft. shows no sign of deterioration; also a letter by Spencer W. Stewart, M. Am. Soc. C. E., in *Engineering News-Record*, September 3, 1925, p. 396. Mr. Stewart states that a record of about seventy-five Ambursen dams built in Canada, in the Northwest, and in the New England States, shows that only three dams have deteriorated. Many of them are subject to minimum temperatures of between 40 and 50° below zero Fahr.; the oldest of these dams was built 22 years ago.

‖ *Proceedings*, Am. Soc. C. E., September, 1925, Papers and Discussions, p. 1324.

it is quite immaterial whether this is accomplished by considering uplift or by lowering the maximum permissible stress.

The authors also state:*

"It is not believed that stresses can be induced in the gravity section by pressures on the upper 30 ft. of the dam, because such pressures will be transmitted direct to the buttresses."

This may be an important question, however, and therefore well worth investigating, rather than leaving it to an offhand assumption. The multiple-arch dam is a much more elastic structure than a gravity dam and the writer's guess would be that the gravity dam might share to a considerable extent the load on the upper 30 ft. of the multiple-arch dam. Since a gravity dam—even the full section having a base width in excess of 81%—has a rather small factor of safety, an additional load applied at the crest and for which no allowance has been made might be a serious matter.

The writer wonders if any provisions were made to prevent the closing of the drain-pipes from the foundation by frost and what the factor of safety of the gravity dam would be in case these drains should become clogged by ice, or in any other manner, and the dam be subjected to uplift?

In the writer's opinion it is of no importance whether the maximum stress is 70 lb. per sq. in., as stated by the authors, or 170 lb. per sq. in., for this does not affect the safety factor in the least. The question is whether tension can be produced in the up-stream face when the dam is affected by adverse conditions.

J. D. Galloway,† M. Am. Soc. C. E. (by letter).‡—There seems to be no question but that the Gem Lake Dam was designed by Mr. Jorgensen in accordance with standard practice and that it was well built. The character of the contractors would lead to the latter inference. All the usual tests of cement and sand were carried out.

The progressive failure of the structure is clearly presented by the authors. Over the entire length of the dam, in the zone covered by the winter fluctuations of the water surface, the concrete was disintegrated and the dam had to be rebuilt as described. The value of the paper lies in the description of the failure.

To make the proper interpretation of this or any other failure of an engineering structure is seemingly more difficult than to describe the failure. It is of interest to read the various explanations given by different engineers. Practically all point to the quality of the concrete as the reason for the failure, the idea being that it was the internal composition of that material that caused the failure and not some cause independent of the concrete. Mr. Jorgensen thinks the cement itself broke down during the coldest weather. The question immediately arises as to why the cement in the concrete above the highest water did not break down. This concrete was subjected to lower temperatures than that in contact with water on one side. The Agnew Lake Dam did not fail and there the water was not lowered during the winter.

* *Proceedings*, Am. Soc. C. E., September, 1925, Papers and Discussions, p. 1324.
† Cons. Engr., San Francisco, Calif.
‡ Received by the Secretary, December 16, 1925.

The evidence seems to the writer to be most plain and the explanation given by the authors correct. The disintegration of the concrete was due to the freezing of the water which had entered into the concrete from the reservoir. The force developed when water freezes is enormous and will easily disrupt concrete. In fact, the same agency is responsible for the wearing down of dense rock masses such as native granite in regions subject to temperatures below freezing. In the present case the thin concrete of the arches permitted the water in the reservoir to be frozen into the form of a layer of ice on the water face, and in addition allowed to freeze whatever water may have been in the voids and hair cracks of the concrete. It is conceivable that cracks would develop in the concrete when on the water face the temperature was that of the water of the reservoir and on the air face the temperature was that of the air, admittedly far below freezing.

The breaking down of concrete by freezing in dams and other structures is not unknown—in fact, is fairly common. Evidences of this have come under the writer's notice and several instances are cited in this discussion. It is unnecessary to look for the fault in the concrete—the cause is found in the type of dam and the operation of the reservoir. It should be noted that in this region the low-water season may occur in the winter when all ordinary springs and streams are frozen. It is then necessary to draw on the reservoir. During a winter, the water surface will alternately rise and fall.

With the conclusions of Alfred D. Flinn, M. Am. Soc. C. E.* the writer desires to differ in every respect. The danger in engineering is not to overemphasize a failure of this type; the contrary is the case. Usually, the effort is directed toward minimizing, or covering up, or explaining away the failure. It is to the credit of the Nevada-California Power Company that it permitted the authors to set forth in detail the failure as it occurred. In most cases the endeavor of the company manager is to suppress all knowledge of such a failure. It is partly by failures of this type that the Engineering Profession advances. The Gem Lake Dam was a test to destruction of a full-sized dam under operating conditions—a test of the most valuable kind. Tests to destruction are always the best, especially if the progress of the test is observed. No others can yield such valuable results in the form of knowledge. This is the type of test to be carried out on the Stevenson Experimental Dam near Fresno, Calif.

Again, little if any information would be gained by counting the dams that have failed and those that have stood. Engineering is not a matter of majorities, one way or the other. For example, there was a time in the United States and elsewhere, when the compression members of metal bridges were made of cast iron. Many of them held together, but some did not. It took the crowning disaster at Ashtabula, Ohio, to teach engineers that cast iron was an improper material for the vital members of a bridge. Notwithstanding the numerous bridges of that type then standing, engineers eliminated cast iron. Too much emphasis cannot be given the Gem Lake failure. It teaches that under such severe climatic and operating conditions, the multiple-arch dam with its relatively thin concrete, should not be used.

* *Proceedings*, Am. Soc. C. E., December, 1925, Papers and Discussions, p. 2015.

The writer also wishes to call attention to what is believed to be an unfortunate tendency among engineers and is exemplified in the present instance. As soon as a concrete structure fails, a number of commentators feel that it is necessary to rush to the defense of concrete as such, and endeavor to explain the failure by citing some unusual defect. In this case, some think that the small percentage of clay and silt explains the whole matter. Possibly so, but the evidence is all to the contrary. As a rule, a small percentage of silt renders the concrete more water-tight. Comparison is made between the structure that failed and some ideal concrete that is sometimes obtained in the laboratory and rarely if ever in the structure as built. Engineering structures are built of the materials at hand. There are regions in the West, of volcanic or granite formation, where it is impossible to obtain anything but a medium quality of sand or rock. Construction, even under the best of supervision, is always subject to conditions that make the resulting structure far short of the ideal. This imperfect structure is the one that engineers have to deal with and that sometimes fails—not the ideal one for which everything is perfect. Although always striving for the very best, the Engineering Profession should design in the light of information obtained in actual practice as to just what degree of perfection may be realized in the structures built under ordinary conditions.

Mr. Jorgensen undoubtedly feels chagrined over the failure of the dam. The writer believes this to be largely unnecessary. His dam was well designed in accordance with good practice, and it was reasonably well built. It resisted the stresses of the water, but failed gradually under severe climatic conditions coupled with certain operating conditions not usual in many cases.

The value of the failure at Gem Lake will lie largely with those who interpret it correctly. To those who see in it merely another example of a defective type of concrete, to be added to the thousands of such examples already existing, the failure will be of no value. To those engineers who find in the combination of thin concrete arches, severe climate, and rather unusual operating conditions an explanation of the failure, the lesson will be of the utmost value. Had Mr. Jorgensen had before him a similar example when designing the Gem Lake Dam it is conceivable that he would have adopted another type. Therein lies the value of the lesson of this failure.

The authors are to be congratulated on the clear manner in which the account was given and on the conclusions reached.

I. OESTERBLOM,* M. AM. SOC. C. E. (by letter).†—This paper is unusually interesting and raises again a question that is causing anxiety all over the world. Slow disintegration of concrete, substantially as described at the Gem Lake Dam, seems to cause even more trouble and anxiety in the tropics. Naturally, one is, therefore, led to ask: "Is it really the cold that does the damage?" This is usually assumed—it seems so obvious; but the obvious is not always correct and is very often far from correct.

* Care, Truscon Steel Co., Youngstown, Ohio.

† Received by the Secretary, December 21, 1925.

In the case under discussion it may be the cause in part, but there also seems to be evidence of other causes. Under the heading, "History of Operation",* it is stated that: "During the eight years of operation of the structure, very few cracks occurred in it." This does not seem to be the behavior of cold weather which acts on entrapped water, with a great deal of violence and speed. The force is evidently of a slower nature, but is none the less very destructive.

There is further evidence that it was not frost that caused the essential part of the destruction. Referring to the first paragraph under the heading, "Condition of Dam in 1924",† it is stated that in the intervening middle belt, the concrete had become dead and inert, but that a section at the bottom of each arch was little affected. If frost had been the cause, would not the effect have been general?

The subsequently recorded opinion of investigating engineers would seem to be inconsistent. If there was sufficient porosity to permit percolation, the destruction, by necessity, would have taken place during the first winter after construction, but this Mr. Jorgensen, himself, is able to disprove by very competent evidence—his own records—long before any trouble was anticipated. The paper also states‡ that "during this time [two years] no change in the structure was apparent." The concrete evidently was most excellent and of the highest consistency.

Everything seems to point to a more insidious force than mere frost and water. The writer suggests the supplementary compressive forces due to the colloidal expansion of the silica gels that make up the cement.

It would be interesting to know where the cracks really occurred from the viewpoint of stress calculation based on such expansion, in addition to the usual effects of gravity and temperature. Such calculations, compared with the actual records of the cracks, would really be conclusive.

Colloidal expansion is also due to presence of water but in very small quantities only and in such a state, ordinarily, that frost would have no direct effect. Even in the best of concrete this water is present and it is only necessary that the external water should vary—as it did in this case—in order to set up a change in colloidal absorption and subsequent variation of internal stress. It is well known how destructive such variation of stress is; it is also known how slowly it works and that a catastrophe, due to such conditions, sometimes may come without warning and most unexpectedly if special precautions are not taken.

It would seem that the destruction recorded at the Gem Lake Dam is an ideal example of how a most excellent piece of construction is being slowly destroyed by colloidal action. At the same time it offers another opportunity for the engineer to take a decided step forward in technical progress.

Destruction similar to that recorded is general in the tropics even to such extent that in certain parts competent engineers refuse to use concrete for permanent construction. The basis there for the change in colloidal absorp-

* *Proceedings*, Am. Soc. C. E., September, 1925, Papers and Discussions, p. 1312.
† *Loc. cit.*, p. 1320.
‡ *Loc. cit.*, p. 1319.

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* *Proceedings*, Am. Soc. C. E., September, 1925, Papers and Discussions, p. 1312.
† *Loc. cit.*, p. 1320.
‡ *Loc. cit.*, p. 1319.

tion and subsequent stress variation is the monsoonal change, with some months very dry and others very wet. A comparison between conditions and effects in cold and warm climates is interesting, and the matter has been discussed.*

B. C. COLLIER,† M. AM. SOC. C. E. (by letter).‡—The writer has read with great interest the various points brought out in the paper, the discussion regarding the failure of the concrete in Gem Lake Dam to withstand the effects of cold or other disintegrating agents, and especially the statement in the Synopsis§ relative to the necessity of providing a "positive and permanent method of water-proofing concrete."

For several years he has been an earnest advocate of the necessity of proper selection of aggregates in order to insure the life of concrete structures, especially when subjected to water conditions. The statements made by several discussors, to the effect that a very probable cause of the failure was the use of limestone as the aggregate, are fully in accord with his experience. For a number of years he has believed that concrete will not withstand the continued action of water pressures especially when it is in thin slab sections. Examination of a number of reservoir and similar linings seems to prove that the continued passing of minute quantities of water through the pores of the concrete has the effect of a gradual attrition of the cement particles, and the consequent breaking down of the structure.

This discussion has also been of especial interest to the writer, because he was asked, during the period of construction of this dam, to pass opinion on the thickness of the "Gunite" covering to be applied to the up-stream face as a water-proofing medium. At that time knowledge regarding such details was very limited, but the suggestion offered to the effect that at least 1 in. of reinforced "Gunite" should be used, was not accepted as advisable. According to the paper a very thin coat ($\frac{3}{4}$ in. at the bottom and $\frac{1}{4}$ in. at the top) was applied without any reinforcement.

It is, therefore, of particular interest to note what the probable result might have been had a thicker coating of "Gunite" with proper reinforcement been applied. All statements made are to the effect that for a period of almost three years no seepage through the dam was noted, but that subsequently this seepage occurred and gradually increased in volume. In a structure such as this the alternate rise and fall of the water naturally causes severe stresses and a tendency to movement in the up-stream surface. Without question, this stress, continued over a period of time, caused the development of checks or cracks in the thin shell of unreinforced "Gunite", thereby ultimately defeating its original purpose, and allowing the passage of the water through the dense surface covering to the porous concrete beneath. Although, as stated, at the time at which this work was done little knowledge was extant regarding methods of water-proofing with "Gunite", later experiences have indicated that had a heavier coat of properly reinforced "Gunite" been used, it would probably have prevented the seepage of the water and the consequent disintegration of

* *Proceedings*, Institution of Engrs. (India), Yearly Meeting, Delhi, 1925.

† Pres. Cement Gun Company, Inc., Allentown, Pa.

‡ Received by the Secretary, December 23, 1925.

§ *Proceedings*, Am. Soc. C. E., September, 1925, Papers and Discussions, p. 1310.

the concrete whether due to faulty sand, faulty aggregate, porosity, or frost action.

It is interesting to note the suggestion made by some of the discussors that a coating of asphalt on the up-stream face would have resulted in the saving of the structure. In view of the statements recently made to the writer by several engineers in charge of large water-works activities, to the effect that not only is asphalt, when subjected to such conditions, not water-proof, but also that the life of a membrane coating under water does not exceed 15 or 20 years, doubt naturally arises as to the efficacy of such treatment. An engineer of wide experience in testing work recently stated that he had definitely proven that asphalt would absorb moisture.

The writer has long been an ardent believer in the adequacy of Portland cement, when it is properly applied, as a resistant to water action. In order to produce this resistance it is necessary to provide a density of application similar to that of "Gunite". In addition, however, a broad experience in inspection of such structures as this had indicated the necessity of properly reinforcing the "Gunite" slab in order to eliminate the breaking down by "weave" and constant movement; or to insure against the development of checking or cracking from internal stresses. Such authorities as F. E. Turneaure, M. Am. Soc. C. E., have clearly shown that cement structures should always be reinforced against the initial "setting up" stresses, and, in accordance therewith, the writer is now recommending that, for "Gunite" slabs for water-proofing, reinforcement of not less than 0.3 of 1% should always be used, and preferably, under such severe conditions as that in Gem Lake, that the percentage should be as high as 0.5. This percentage of reinforcement placed equally in both directions will also provide an almost positive protection against cracks that will arise from expansion and contraction. With such reinforcement, and with a thickness of approximately 2 in. of "Gunite", experience has shown that the positively water-proof conditions specified by the authors as being necessary will be obtained.

FRED A. NOETZLI,* ASSOC. M. AM. SOC. C. E. (by letter).†—It is well known that certain concretes are subject to deterioration from frost and other actions when they contain a considerable amount of moisture. The deterioration of concrete in sea water is a striking example of this phenomenon. In this case the physical effect of the expansion of the salt crystals which, under certain conditions, may form in the pores of porous concrete in sea water is increased by chemical action. In hydraulic structures subjected to frost conditions the particles of water contained in the concrete will expand considerably when this water is transformed into ice, and thus tend to spall the concrete or weaken it otherwise. Porous concrete is, therefore, apt to suffer considerably from frost action. Dense concrete is known to be practically immune against it.

The authors point out that serious deterioration has taken place in the arches of the Gem Lake Dam where the concrete was subjected to water pres-

* Cons. Hydr. Engr., Los Angeles, Calif.

† Received by the Secretary, December 24, 1925.

sure at the time of low temperatures. On the other hand, M. M. O'Shaughnessy, M. Am. Soc. C. E., has stated* that in the Lake Eleanor multiple-arch dam, built in 1918, at an elevation of 4 660 ft., "there are no indications of wear and tear on this concrete structure, and the location is subject to severe frosts without any noticeable depreciation whatever in the concrete"; further, that, in his opinion, "the failure of the Gem Lake Dam is due not to design of structure, but to defective granite dust and materials being incorporated in the sloppy overwatered concrete during construction."

The writer can testify from personal knowledge that, also, the Bear Valley multiple-arch dam, located at an elevation of 6 700 ft., shows very little, if any, deterioration from frost action.

The Gem Lake Dam was built in 1915 and 1916. The Bear Valley and Lake Eleanor multiple-arch dams were constructed in 1912 and 1918, respectively. The Gem Lake Dam suffered from frost action, the other two dams are apparently as strong as at any time in their history, although the climatic conditions are similar at all three sites.

What was wrong, then, with the concrete of the Gem Lake Dam? The authors state that the upper 30 ft. of the existing arches and the buttresses supporting them appeared to be in perfect condition. Only that part of the arches which was below the reservoir level during the severe winter conditions was damaged. The temperature variations in the upper arches and the buttresses were at least as great as for the lower arches, and probably greater. It was, therefore, apparently not the low temperature itself that damaged the lower arches, but evidently the water which had been filtered into the concrete and which expanded when it was transformed into ice in freezing weather. Similar deterioration of concrete has been observed in many structures exposed to water and frost, in canal head-works as well as in some gravity dams many feet in thickness. In most of these cases, deterioration could easily be traced to porous concrete, or unsuitable aggregate, or both.

For the construction of the Gem Lake Dam, a lake sand, containing 3½% of clay and 1% of dirt, was mixed with the sand from the rock crusher in the proportion of about 75% of lake sand to 25% of crushed rock sand. It is not stated whether or not this sand was properly graded. The fact that compression tests on specimens made during construction showed an average crushing strength of the 1:2:4 concrete of only 900 lb. per sq. in. at the age of 14 days would indicate that the quality of the sand used was not at all what it should have been for this type of structure. Chemical decomposition of certain ingredients of the granite dust of the "crusher run" may also have been one of the reasons for the troubles with the dam.

The authors deserve credit for emphasizing the necessity of seriously considering frost action when designing dams for cold climates. The method of repair adopted for the Gem Lake Dam is about as radical and expensive as could be devised, but it may have been necessary in view of the advanced state of deterioration.

The writer believes that a satisfactory water-proofing of the arches, as originally attempted by the owners of the dam, could have been obtained

* In a letter to the Editor of *Engineering News-Record*, July 30, 1925.

in the following manner: At the time when the concrete of the arches showed signs of deterioration, the weak spots, leaky construction joints, etc., should have been cut out as far as practicable and re-filled, for instance, by means of a cement gun. Then a four-ply water-proof membrane of a Malthoid or Tartex product might have been applied in successive layers on the up-stream face of the arches, and a protection slab of concrete, from 12 to 18 in. in thickness, cast against the membrane. The four-ply water-proof membrane would have protected the old arches from infiltrations of water and thus have stopped further deterioration. The concrete protection slab, besides preventing injury to the membrane, would have added materially to the safety of the dam. A dense concrete such as was used successfully for the Lake Eleanor multiple-arch dam, referred to previously, would have insured the permanency of the protection slab. The cost of this concrete slab would have been relatively small as no forms would have been required for the under side of the slab. The water-tight membrane of the character described would have cost about 15 cents per sq. ft. in place. The quantity of concrete for a 12-in. protection slab from the bed-rock up to an elevation of 30 ft. below the crest of the dam, would have been approximately 1 250 cu. yd. This compares with the 12 000 cu. yd. used for the method of repair adopted by the owners of the dam. The multiple-arch type of dam has the advantage of offering easy accessibility to all parts of the structure. Repairs may be made, therefore, before the danger point has been reached.

In what follows, the writer wishes to make a few suggestions which may be helpful for avoiding in future thin dams some of the difficulties that have arisen in connection with the Gem Lake Dam.

The writer agrees with the authors that an impervious concrete will prevent the deterioration of concrete subjected to water pressure and extreme low temperatures. A careful selection and the proper grading of sand and gravel or crushed rock will go a long way toward accomplishing this end. A somewhat richer mixture than the standard 1 : 2 : 4 proportion, say, about 1 : 1.8 : 3.5, will furnish a dense and strong concrete at a comparatively small increase in cost per cubic yard. Naturally, proper care in mixing and placing the concrete is also of great importance.

The arches of multiple-arch dams should be designed so that for shrinkage, rib-shortening, and maximum variations of temperature the tension stresses in the arches are safely within the tensile strength of the concrete. A liberal quantity of reinforcing steel should be added for additional safety. Special attention should be given to the preparation of construction joints between successive horizontal layers of concrete. Leakage is most likely to occur in joints that have not been properly cleaned of laitance and foreign matter. Attack by frost would inevitably start sooner or later in such joints, and the deterioration of the concrete would continue progressively. A 4-in. strip of galvanized iron, placed from 3 to 4 in. from the up-stream side of the arches, across all horizontal as well as vertical construction joints, will prevent leakage through the joints.

In important structures the entire up-stream face of the arches should be covered with a layer of properly reinforced "Gunite" from 1 in. to 2 in. thick.

Thin layers of unreinforced "Gunite" such as was used on the Gem Lake Dam, are not effective very long.

In spite of the partial disintegration of the Gem Lake Dam, the writer firmly believes that safe and permanent structures of the thin arch and multiple-arch type may be built in locations subject to severe weather conditions. A proper design of the arches, suitable, well-graded concrete materials, and careful mixing and placing of the concrete will result in structures which will be safe, water-tight, and permanent. A considerable number of such dams were built years ago and are satisfactory in every way. A few others, like the Gem Lake Dam, have given trouble for reasons which are fairly well understood by this time and can easily be avoided in future constructions.

A multiple-arch dam that will be 160 ft. high and more than ½ mile long is now under construction in the high Sierras at an elevation of 7 200 ft. The multiple-arch type was selected for this structure after careful investigation of the probable causes that may have been responsible for the troubles on Gem Lake Dam.

The authors seem to favor dams of the rock-fill type for locations where extremely low temperatures may occur. Such rock-fill dams naturally would have to depend for water-tightness on a concrete core wall or on a concrete face slab. A loose rock-fill is relatively porous and permits of more or less free air circulation, including "chimney effect", that is, allowing the warmer and lighter air in the interior of the mass to rise and escape at the top of the dam, thus drawing in the cold outside air through the lower porous parts of the fill. The thin slab of a concrete core or face wall of such a rock-fill dam, therefore, would undoubtedly be subjected to some frost action, which, in case of porous or otherwise unsuitable concrete, sooner or later would unquestionably lead to progressive deterioration of the concrete. Inasmuch as there would be no simple way of observing such deterioration of the most vital part of a rock-fill dam, a "blow-out" of a weak spot and a possible failure of the structure might occur in time.

If on Gem Lake a rock-fill dam had been constructed, and if for the core or face wall the same kind of aggregate had been used with the same class of workmanship as for the multiple-arch dam, it is possible that the authors would have had occasion to describe the failure of a rock-fill dam instead of giving an interesting account of the successful repairs of a multiple-arch dam.

THE IMPROVED VENTURI FLUME

Discussion*

By Messrs. Julian Hinds, E. A. Jacob, and J. C. Stevens.

Julian Hinds,† M. Am. Soc. C. E. (by letter).‡—This is an important contribution to engineering knowledge on the subject of water measurement. The open Venturi flume has come to be recognized as a valuable device because of its freedom from interference by silt accumulations and because of the small head required for its operation.

As originally contemplated, the Venturi flume requires the reading of two gauges, as in the case of the closed Venturi tube. Because of the variable areas of the water prism at points of measurement, diagrams and tables for the original device are more complicated than for the closed Venturi meter. It is now proposed to eliminate one of the gauges and to simplify the utilization of the device, putting it on an equality with the sharp-crested weir so far as reading the discharge is concerned, without seriously interfering with the advantages of the original type.

The writer is in full accord with the proposal to develop a one-gauge Venturi flume, and is of the opinion that the type of structure tested by the author can be satisfactorily developed.

However, it is believed that the proposed details can be improved. The discharge formula presented by the author is entirely empirical, and is applicable only to structures of the precise form tested. By slightly altering the structural details it is believed that the device can be made to operate in accordance with a rational theory. The writer considers this desirable, although the resulting formula may be as difficult to use as the empirical formula. The proposed flume takes partial advantage of the controlling effect of a point of critical depth in open-channel flow. By taking full advantage of this effect the flow formula can be rationalized.

If a flow of water is required to pass through the flume shown in Fig. 6, the floor of the flume being level from *A* to *D* and moderately steep down stream from *E*, then, at some point between *D* and *E*, flow will take place under a minimum energy head and the depth will be critical. The critical depth will occur at the point where the slope of the bottom of the channel is just sufficient to overcome friction due to flow at critical depth, which point is called the "control point." The corresponding cross-section of the canal may be called the "control section."

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* Discussion of the paper by Ralph L. Parshall, Affiliate, Am. Soc. C. E., continued from January, 1926, *Proceedings*.

† Engr., U. S. Bureau of Reclamation, Denver, Colo.

‡ Received by the Secretary, November 27, 1925.

The theory of the critical depth and the point of control has been thoroughly discussed in recent years before the Society and in technical publications, and need not be referred to here further than to call attention to the fact that it assumes normal hydrostatic pressure throughout the depth of the stream. At points of irregularity, such as sharp corners in the floor or sides of the channel, or at the outfall end of a flume discharging freely into the air, the hydrostatic pressure in the stream departs from normal, and the energy equation is correspondingly disturbed. Usually the critical depth is not situated exactly at such points, although it is often assumed to be.

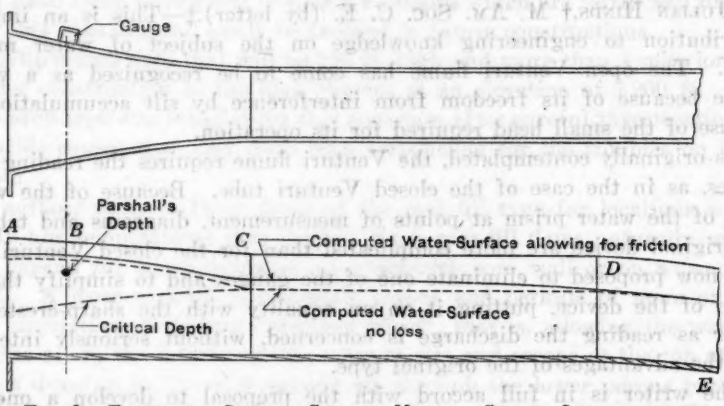


FIG. 6.—ELONGATED CONTROL-SECTION METER. CURVED INTERSECTIONS.

If irregularities of this kind are avoided, a point of control always exists where an open channel changes, within certain limits, from a comparatively flat slope to a comparatively steep slope. At the point of control many of the uncertain factors normally affecting flow, such as unknown friction coefficients, velocity of approach, etc., are eliminated, and the depth bears a definite relation to the discharge. Thus, by introducing into a canal a proper change in grade, a point of definite stage, for any given flow, is established. It is not necessary that the steep slope to the right of E in Fig. 6 continue for any great distance, provided the elevation of the back-water below, after allowing for recovery of velocity head, is low enough to prevent submergence of the point of control.

So far as the writer knows, no attempt has been made to utilize the directly measured critical depth for determining discharge, although the relation between this depth and the flow is simple. This is probably because the exact location of the control is likely to be uncertain. The depth near the critical point is subject to rapid variation and there is a lack of confidence in measurements taken at the theoretical point of control. A flume such as that shown in Fig. 6, 2.00 ft. wide, will flow 2.00 ft. deep at the point of control with a discharge of 32.08 sec-ft. With Kutter's $n = 0.013$, the friction slope for the critical depth is approximately 0.008. The friction loss in the first 0.25 ft. above the control, therefore, will be about 0.002 ft.

To build up the energy head needed to compensate for this loss requires an increase in depth of approximately 0.05 ft. Therefore, if the floor of the flume is level above the control, the water surface will rise more than 0.5 in. in the first 3 in. of channel, due to the flatness of the energy curve near the point of critical depth. For shorter distances the rate of rise is even more pronounced, and a slight error in the selection of the point of measurement will appreciably affect the measured depth. A curved bottom for a short distance above the control tends to compensate for the disadvantageous shape of the energy curve, and it may eventually be found possible, with a proper type of structure, to use the measured critical depth for the determination of discharge. Such a plan would have the distinct advantage of being free from the influence of changing approach conditions.

With the gauge located in the inlet section as advocated by the author, the theoretical relation between depth and discharge may be computed from Bernoulli's theorem, making allowance for friction and transition losses between the point of measurement and the control. If the length of the structure is great, or if the transition is imperfect, the uncertainty as to the total value of the losses is likely to be considerably greater than if the depth is measured at the point of control. However, the effect on the computed flow of an error in determining the loss is less marked with the gauge in the inlet. The gauge should be set at a point where the channel will always be swept clean by the velocity, to avoid interference by accumulations of silt.

Two water surface curves are shown in Fig. 6, one for frictionless flow and one allowing for friction due to Kutter's $n = 0.013$. Transition losses are disregarded. The divergence of the two lines indicates that with carefully rounded corners of the type shown, and with a long radius vertical curve from D to E, the head may be safely measured at the control.

The gauge height for the assumed flow, computed from the author's formula,* is shown at A (Fig. 6), and is appreciably lower than that computed from the control theory. This discrepancy is explained by a study of the flume shown in Fig. 7, which is identical with that of Fig. 6, except that all curves in the side-walls and bottom are omitted. The inlet is identical with that proposed by the author for a 24-in. flume. Because of the angular change in the slope at I, the stream will leave, or tend to leave, the bottom. As a result there will be a reduction in the hydrostatic pressure, and an alteration of flow conditions, preventing the formation of a control at I. Passing up stream from I, normal hydrostatic conditions are no doubt gradually approached, and if the flume is sufficiently long a control will exist at some point. If a control does exist, the flow may be determined from a gauge reading at G, the only effect of the irregularity at I being the introduction of further uncertainty as to the head lost in friction between the gauge and the point of control. The contractions at H, Fig. 7, introduce eddy losses of unknown value which probably affect the gauge reading at G more seriously than do the irregularities at I.

The throat section, or level, parallel part of the flume in Fig. 7 is purposely made long, to avoid the possibility of overlapping the effects of disturbances at

*Proceedings, Am. Soc. C. E., September, 1925, Papers and Discussions, p. 1342.

the two ends. If the distance, *H I*, is gradually reduced the point of control will approach the disturbed region below *H* and will eventually pass above *H*. An entirely new set of conditions will thus be set up, and if the inlet transition is blunt, weir discharge will be approached. If the taper of the inlet is gradual a control probably will be established a short distance up stream from the junction between the inlet and the throat. The control thus coming at a point where the flume is wider than in the throat, the relation of the gauge reading at *G* to the flow will be altered.

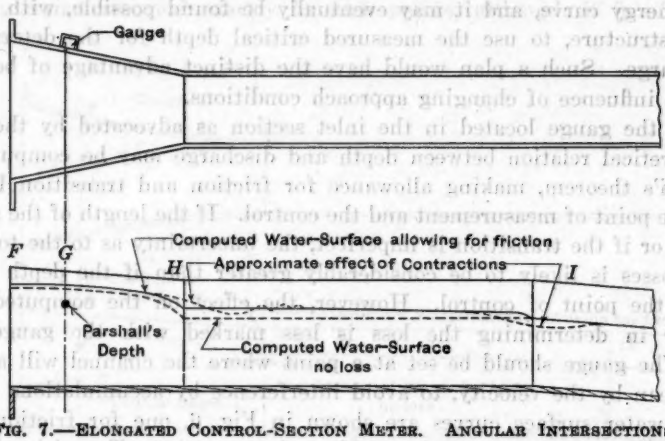


FIG. 7.—ELONGATED CONTROL-SECTION METER. ANGULAR INTERSECTIONS.

This is believed to be the situation existing in the author's device. The throat of the structure is not lengthened, as claimed, but is shortened to a zero length by the vertical expansion beginning immediately below the end of the converging inlet.

The distance from the end of the transition up stream to the point of control is probably influenced by the angle of break in the grade line. The relation between the depth at the gauge and the discharge depends on the width of the channel at the point of control, which is dependent on the location of the control and on the rate of flare of the inlet. The author's formula is, therefore, applicable only to flumes constructed in strict conformity with the structures tested.

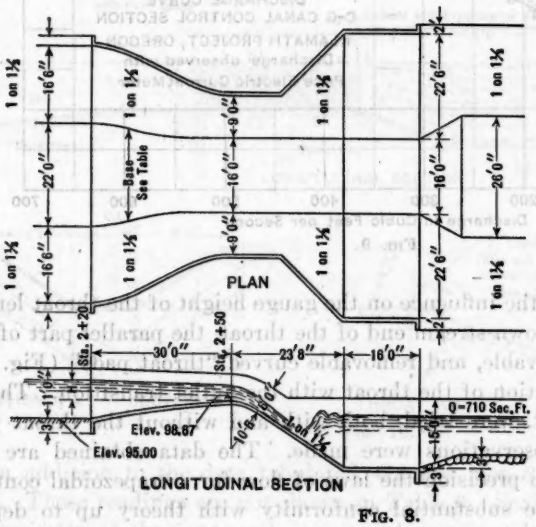
By making the floor of the throat section of the flume level, extending the throat sufficiently to avoid overlapping the effects of angularity at the two ends, and providing sufficient fall to prevent flooding the throat, the device can be made to follow the theoretical law for flow through a point of critical depth. This alteration brings the structure back to the form of the original Venturi flume, which the writer believes to be a better structure fundamentally than the one now proposed.

The U. S. Bureau of Reclamation has developed a type of measuring device designed to operate on the control-section principle. To facilitate construction in concrete, without the use of forms, the throat and approach walls are given a 1 on 1.5 slope. A discussion* by the writer regarding this structure and its

* "Venturi Flume Data Throws Light upon Control Weir," *Engineering News-Record*, December 23, 1920, p. 1223.

operation was published in 1920. The bearing of Venturi flume tests made at Cornell University on the control-section theory was discussed at the same time.

The first experiments by the Bureau of Reclamation on the trapezoidal control section were made at a structure on the C-G Cut-Off Canal on the Klamath Project (Fig. 8), in Oregon, and have been described in a paper* by J. L. Savage, M. Am. Soc. C. E. A number of additional measurements have been made at this structure. All the ratings made to date are shown on Fig. 9. The discharge measurements were made with a current meter. Field measurements were made by Messrs. Allen L. Darr† and H. K. Smith.‡ The agreement between the current-meter discharges and the theoretical discharge of the control section is good. A single test made on a similar structure on the main canal of the King Hill Project, in Idaho, showed practically exact agreement with the theoretical discharge.



BASE DATA

Sta.	Elev.	Width
2+20	95.00	22.0
+22	95.25	21.9
+24	95.60	21.8
+26	96.08	21.0
+28	96.60	20.0
+30	97.00	18.8
+32	97.30	17.8
+34	97.57	17.0
+36	97.84	16.5
+38	98.04	16.1
+40	98.24	16.0
+42	98.41	16.0
+44	98.53	16.0
+46	98.61	16.0
+48	98.66	16.0
+50	98.67	16.0

CONTROL SECTION
C-G CUTOFF CANAL

FIG. 8.

In the fall of 1923, a 24-in. trapezoidal control-section structure was installed at the Hydraulic Laboratory of the Bureau of Reclamation, at Boise, Idaho, and a series of tests were run by Mr. W. G. Steward.‡ The structure was built of wood, the inlet being warped to conform as nearly as practicable to the lines of the standard concrete device. Timber construction was adopted to facilitate remodeling during the progress of the tests. It is not contemplated that wood will be used for permanent installations of the warped type. Details of the structure as installed are shown in Fig. 10. Results of the ratings are shown in Table 6 and Fig 11. The position of the scales and gauges and the hydraulics of a typical run are shown on Fig. 6.

* "Discharge Measurements for Control-Section Weir," *Engineering News-Record*, May 15, 1924, p. 842.

† Asst. Engr., U. S. Bureau of Reclamation, Klamath Falls, Ore.

‡ Asst. Engr., U. S. Bureau of Reclamation, Boise, Idaho.

Gauge readings were made with hook gauges in stilling-wells, located as shown on Fig. 12. The flat scale at the upper end of the inlet structure was countersunk flush with the surface of the wall. The structure was constructed and maintained accurate to the theoretical dimensions within 0.01 ft. The discharges were measured volumetrically in carefully calibrated concrete-lined measuring tanks.

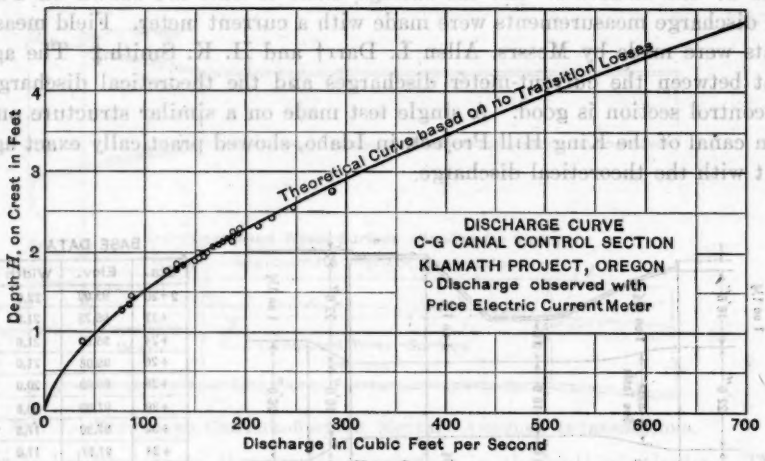


FIG. 9.

In order to determine the influence on the gauge height of the throat length and of curvature at the down-stream end of the throat, the parallel part of the structure was made removable, and removable curved "throat pads" (Fig. 10) were provided at the junction of the throat with the outlet transition. Throat lengths from zero to 3 ft. were tried, both with and without the throat pad. Eighty-seven separate observations were made. The data obtained are not sufficient to establish with precision the laws of flow for a trapezoidal control-section, but they indicate substantial conformity with theory up to depths approximately equal to the throat length.

During the progress of the tests a slight erosion of the banks occurred above the up-stream head-wall, causing a disturbance opposite Gauge No. 1 and the scale. To prevent this condition the side-walls were extended as indicated by dotted lines on the plan view, Fig. 10. These extensions were maintained in place from Series 5, Run 2, to Series 11, Run 6. For Series 12 to 15, inclusive, the extensions were removed and the banks carefully rip-rapped.

The theoretical discharge curve shown in Fig. 11 is based on the assumption of no entrance or friction loss, and is determined by making the gauge depth for a given flow such that the energy gradient at the gauge is equal to the minimum energy gradient at the control.

The general conformity between observed and computed discharges indicates that friction and inlet losses are small. Increased friction in the long throat partly explains the apparent reduction in discharge for Series 15. The computed theoretical discharges shown in Table 6 were scaled from an enlargement

of the theoretical curve shown in Fig. 11. All notes shown under the heading, "Remarks", in Table 6 are copied from the original field notes.

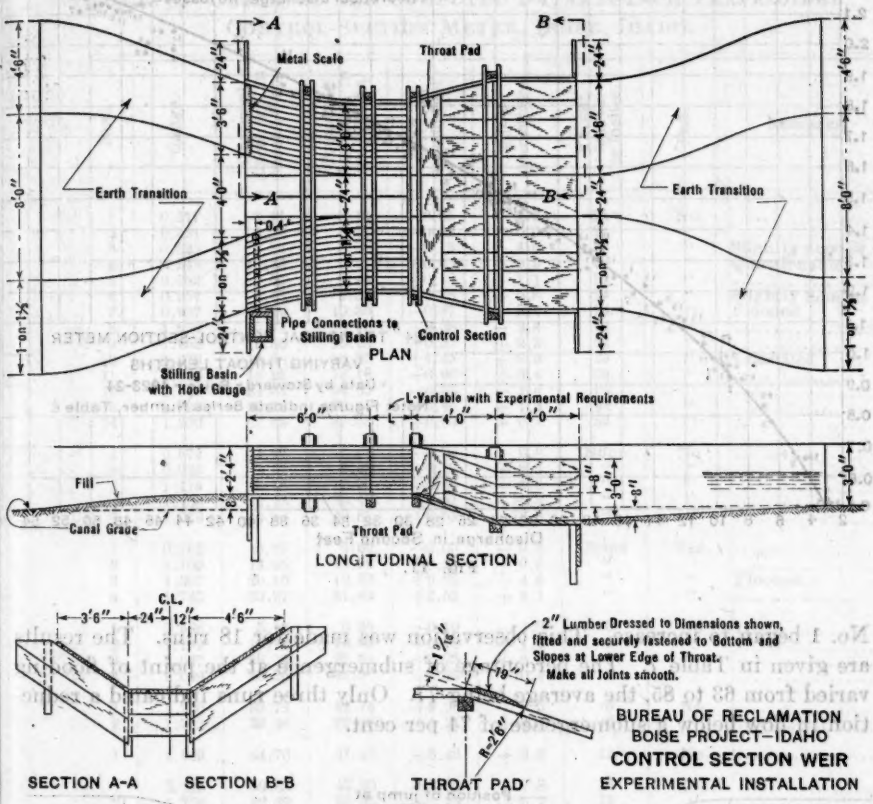


FIG. 10.

In addition to the data tabulated, Gauges Nos. 2 and 3 were read for all runs. These readings are not shown in Table 6. Scale No. 1 was also read for the majority of the runs, and, in general, gave results reasonably close to the theoretical. The scale was originally installed as shown in plan in Fig. 10. For this position of the scale the readings were not the same on the two edges, which caused some confusion. Scales located at other points were tried out later, but yielded no results of particular value. The impression was gained that a scale, properly set, is sufficiently accurate for field use for irrigation purposes.

Fig. 12 shows the hydraulic conditions existing at the time of Run 4, Series 3, with a discharge of 33.97 sec.-ft. The condition shown is typical for the structure. A standing wave was maintained below the throat for all non-submerged flows. By means of a check located down stream from Gauge No. 3, back-water could be caused to flood the control. The point of flooding was determined by slowly increasing the depth at Gauge No. 3, with constant flow, and observing the point at which the depth at Gauge

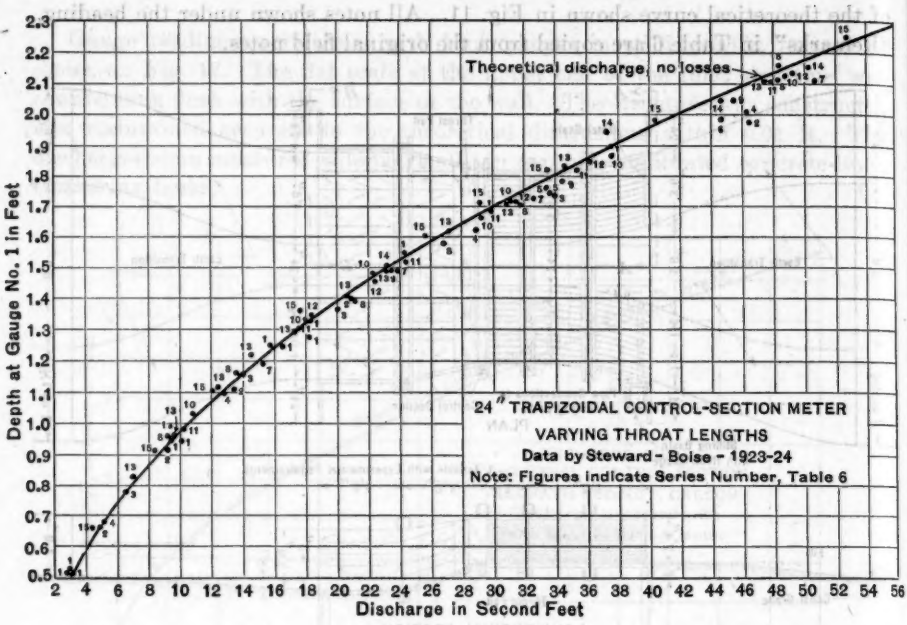

Theoretical discharge, no losses
24" TRAPIZOIDAL CONTROL-SECTION METER
VARYING THROAT LENGTHS
Data by Steward - Boise - 1923-24
Note: Figures indicate Series Number, Table 6
Depth at Gauge No. 1 in Feet
Discharge in Second Feet

FIG. 11.

No. 1 began to increase. This observation was made for 18 runs. The results are given in Table 7. The percentage of submergence at the point of flooding varied from 63 to 85, the average being 77. Only three runs indicated a reduction in flow below a submergence of 74 per cent.

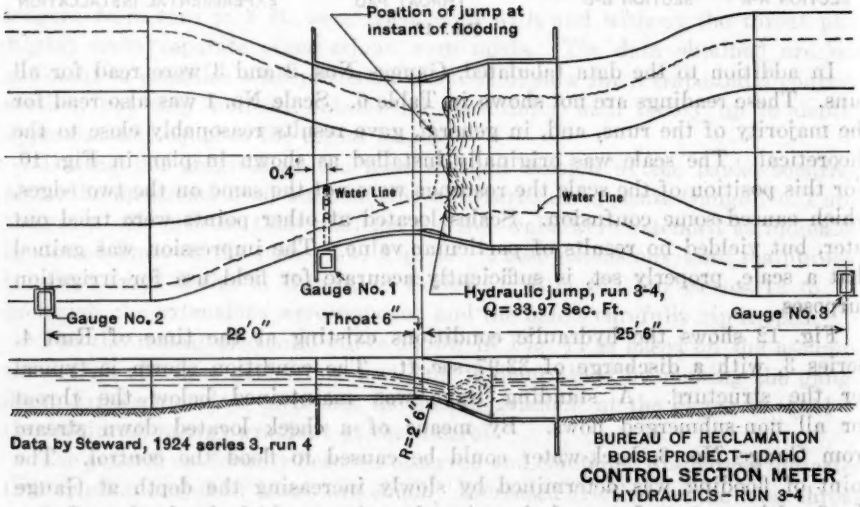


FIG. 12.—TYPICAL RUN, EXPERIMENTAL CONTROL SECTION METER, BOISE, IDAHO.

TABLE 6.—OBSERVED AND COMPUTED DATA, 24-INCH TRAPEZOIDAL CONTROL-SECTION METER, BOISE, IDAHO.

Series.	Run.	Gauge.	Observed discharge, in second-feet.	Computed discharge, in second-feet.	Error, in second-feet.	Error, percentage.	Throat length, in inches.	Pad.	Remarks.
1	1	0.517	3.11	3.16	−0.05	− 1.6	24	No	
	2	0.521	3.19	3.21	−0.02	− 0.6	24	"	
	3	0.525	3.15	3.28	−0.13	− 4.0	24	"	Slightly flooded
	4	0.949	10.06	9.41	+0.65	+ 6.9	24	"	Questionable
	5	0.952	9.55	9.45	+0.10	+ 1.1	24	"	
	6	0.954	9.43	9.51	−0.08	− 0.8	24	"	Slightly flooded
	7	0.997	9.49	10.36	−0.87	− 8.4	24	"	Flooded
	8	1.250	15.74	16.12	−0.38	− 2.4	24	"	
	9	1.252	16.55	16.18	+0.37	+ 2.3	24	"	
	10	1.278	18.28	16.85	+1.43	+ 8.5	24	"	Unsettled
	11	1.326	18.26	18.18	+0.08	+ 0.4	24	"	
	12	1.548	24.34	24.69	−0.35	− 1.4	24	"	
	13	1.690	29.94	29.48	+0.46	+ 1.6	24	"	
	14	1.900	37.65	37.61	+0.04	+ 0.1	24	"	
2	1	0.672	4.97	5.01	−0.04	− 0.8	None	No	
	2	0.923	9.22	8.93	+0.29	+ 3.3	"	"	
	3	1.116	13.46	12.90	+0.56	+ 4.3	"	"	
	4	1.408	22.73	20.42	+2.31	+11.3	"	"	
	5	1.979	46.50	40.85	+5.65	+13.8	"	"	
3	1	0.782	6.57	6.60	−0.03	− 0.5	None	Yes	
	2	1.160	14.05	13.92	+0.13	+ 0.9	"	"	
	3	1.367	20.10	19.27	+0.83	+ 4.3	"	"	Flooded
	4	1.745	33.97	31.42	+2.55	+ 8.1	"	"	
4	1	0.688	5.16	5.26	−0.10	− 1.9	6	No	
	2	1.102	12.78	12.63	+0.15	+ 1.2	6	"	
	3	1.627	29.03	27.30	+1.73	+ 6.4	6	"	
	4	2.009	46.25	41.90	+4.35	+10.4	6	"	
5	1	1.750	33.78	31.73	+2.05	+ 6.5	6	No	
	2	1.762	33.46	32.15	+1.31	+ 4.1	6	"	
6	1	1.989	44.70	41.28	+3.42	+ 8.3	12	No	
7	1	2.115	50.65	47.00	+3.65	+ 7.8	12	Yes	
	2	1.728	32.60	30.85	+1.75	+ 5.7	12	"	
	3	1.492	23.91	22.85	+1.06	+ 4.6	12	"	
	4	1.200	15.21	14.90	+0.31	+ 2.1	12	"	
8	1	0.957	9.44	9.58	−0.14	− 1.5	18	No	
	2	1.162	13.86	13.95	−0.09	− 0.6	18	"	
	3	1.401	20.87	20.28	+0.59	+ 2.9	18	"	
	4	1.583	26.86	25.83	+1.03	+ 4.0	18	"	
	5	1.710	31.63	30.20	+1.43	+ 4.7	18	"	
	6	2.156	48.45	48.92	−0.47	− 1.0	18	"	
	7	2.114	48.20	46.95	+1.25	+ 2.7	18	"	
9	1	1.497	23.44	22.95	+0.49	+ 2.1	18	Yes	
	2	1.790	34.72	33.25	+1.47	+ 4.4	18	"	
	3	2.048	45.50	43.95	+1.55	+ 3.5	18	"	
10	1	1.663	29.28	28.60	+0.68	+ 2.4	24	No	
	2	2.127	48.80	47.55	+1.25	+ 2.6	24	"	
	3	1.030	10.93	11.05	−0.12	− 1.1	24	"	
	4	1.325	18.03	18.15	−0.12	− 0.7	24	"	
	5	1.483	22.37	23.60	−0.23	− 1.0	24	"	
	6	1.723	31.10	30.70	+0.40	+ 1.3	24	"	
	7	1.876	37.66	36.61	+1.05	+ 2.9	24	"	
	8	2.156	50.09	49.00	+1.09	+ 2.2	24	"	
11	1	0.988	10.30	10.19	+0.11	+ 1.1	24	Yes	
	2	1.307	17.68	17.62	+0.06	+ 0.3	24	"	
	3	1.523	24.41	23.90	+0.51	+ 2.1	24	"	
	4	1.090	29.93	29.45	+0.48	+ 1.6	24	"	
	5	1.823	35.43	34.58	+0.85	+ 2.5	24	"	
	6	2.108	47.93	46.65	+1.28	+ 2.7	24	"	

TABLE 6.—(*Continued*).

Series.	Run.	Gauge.	Observed discharge, in second-feet.	Computed discharge, in second-feet.	Error, in second-feet.	Error, percentage.	Throat length, in inches.	Pad.	Remarks.
12	1	1.345	18.37	18.68	−0.31	− 1.7	24	Yes	
	2	1.465	22.43	22.02	+0.41	+ 1.9	24	"	
	3	1.719	31.55	30.48	+1.07	+ 3.5	24	"	
	4	1.852	36.25	35.70	+0.55	+ 1.5	24	"	
	5	2.136	49.21	48.00	+1.21	+ 2.5	24	"	
13	1	0.962	9.67	9.67	0.00	0.0	24	Yes	
	2	0.833	7.12	7.40	−0.28	− 3.8	24	"	
	3	1.113	12.47	12.84	−0.37	− 2.9	24	"	
	4	1.222	14.54	15.45	−0.91	− 5.9	24	"	
	5	1.293	17.04	17.28	−0.24	− 1.4	24	"	
	6	1.415	20.72	20.65	+0.07	+ 0.3	24	"	
	7	1.497	23.32	22.95	+0.37	+ 1.6	24	"	
	8	1.622	27.21	27.18	+0.03	+ 0.1	24	"	
	9	1.713	31.05	30.30	+0.75	+ 2.5	24	"	
	10	1.828	34.62	34.75	−0.13	− 0.4	24	"	
	11	2.113	47.66	46.92	+0.74	+ 1.6	24	"	
14	1	2.153	50.23	48.84	+1.39	+ 2.8	24	No	
	2	2.050	44.63	44.04	+0.59	+ 1.3	24	"	
	3	1.948	37.30	39.60	−2.30	− 5.8	24	"	Questionable
	4	1.506	23.25	23.28	−0.03	− 0.1	24	"	
15	1	0.664	4.48	4.94	−0.46	− 9.4	36	No	
	2	0.923	8.41	8.94	−0.53	− 5.9	36	"	
	3	1.106	12.08	12.70	−0.62	− 4.9	36	"	
	4	1.364	17.77	19.20	−1.43	− 7.4	36	"	
	5	1.605	25.68	26.58	−0.90	− 3.4	36	"	
	6	1.711	29.24	30.24	−1.00	− 3.3	36	"	
	7	1.822	33.57	34.55	−0.98	− 2.8	36	"	
	8	1.978	40.58	40.80	−0.22	− 0.5	36	"	
	9	2.245	52.85	53.35	−0.50	− 0.9	36	"	

While making the submergence test, the standing wave was closely watched to determine, if possible, the visual evidence of the change from free to submerged flow. Submergence appears to begin when the wave reaches approximately the location of the control. The control in Fig. 12 might be expected to come at about the point shown for the up-stream position of the jump. Run 8, Series 10, was made with a 24-in. throat without a throat pad, and, following the theory outlined in the first part of this discussion, the control might be expected to be a short distance above the down-stream end of the throat. Flooding began when the wave had moved 6 in. back into the throat. Run 1, Series 3, was made with the throat pad, but with a zero throat length. Flooding began with the wave slightly farther up stream than the position shown for flooding in Fig. 12. The characteristic appearance of the wave just before submergence is shown in Fig. 13, and a view of the control-section weir running at full capacity is shown in Fig. 14. After submergence the wave usually changes to a series of ripples, which may be practically as rough as the last stage of the wave, but which have a distinctly different appearance. It is believed that with a little development the change from submerged to non-submerged flow can be made clearly visible.

No attempt was made to rate the structure for submerged flow, since under such conditions the device becomes a true Venturi flume, for which many ratings have been made.

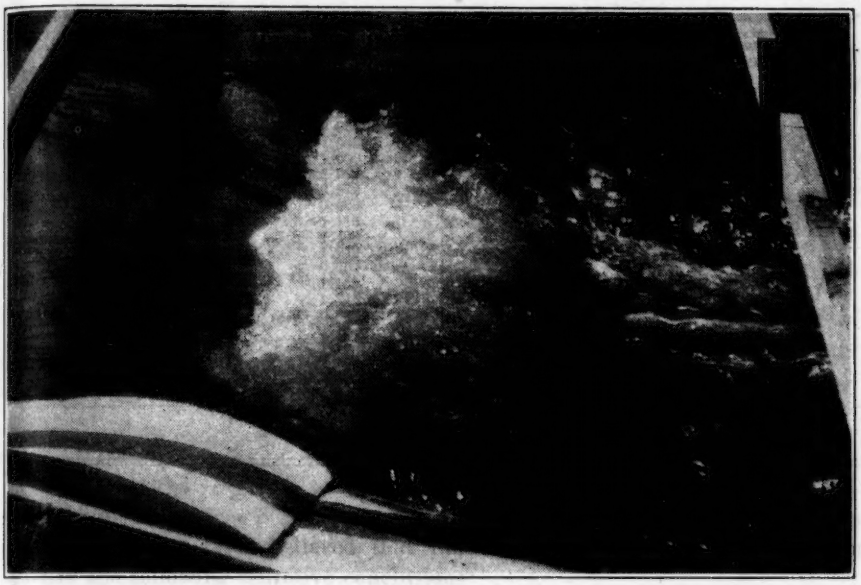


FIG. 13.—CONTROL-SECTION METER AT BOISE, IDAHO, SHOWING HYDRAULIC JUMP.

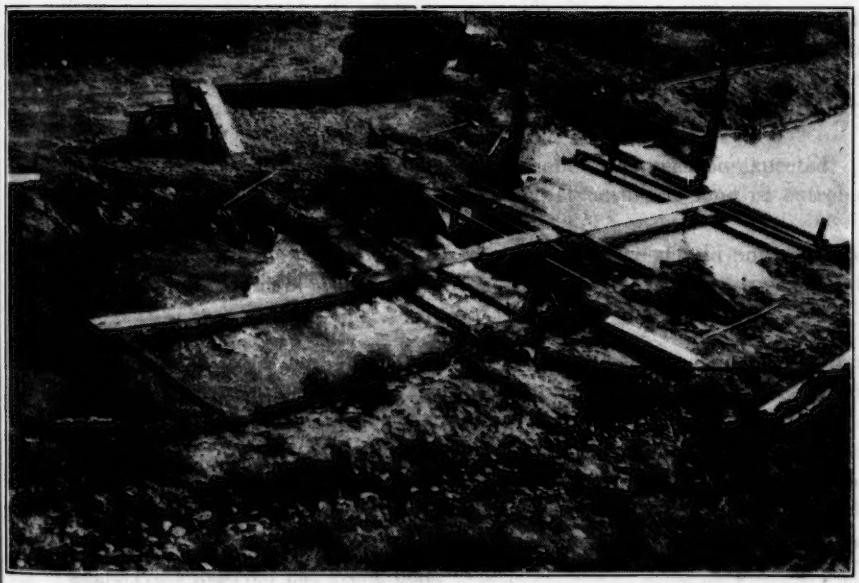
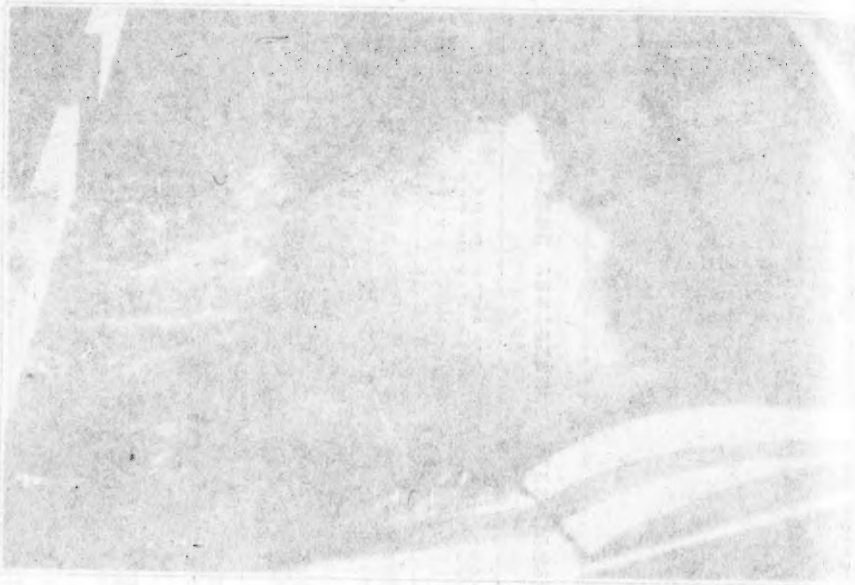


FIG. 14.—CONTROL-SECTION WEIR AT BOISE, IDAHO, RUNNING AT FULL CAPACITY.



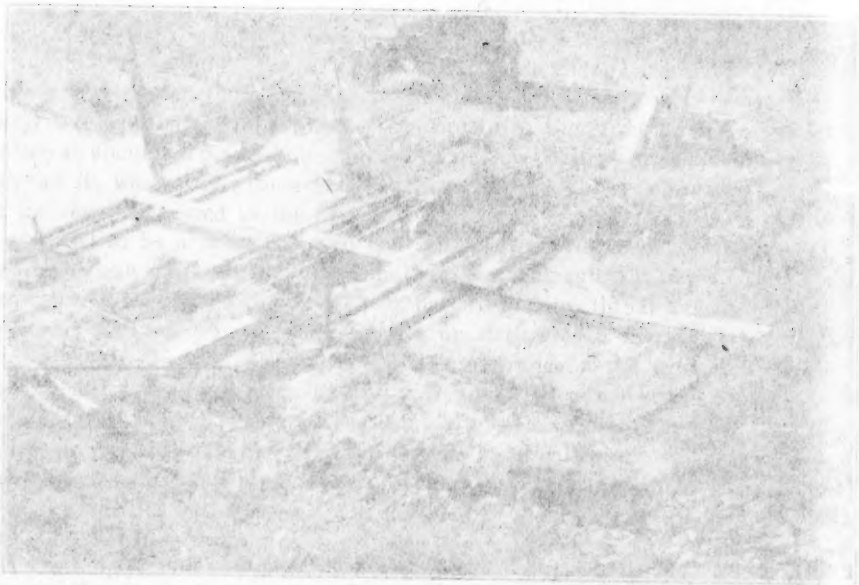


TABLE 7.—SUBMERGENCE REQUIRED FOR FLOODING, 24-INCH TRAPEZOIDAL CONTROL-SECTION METER, BOISE, IDAHO.

Series.	Run.	Reading at Gauge No. 1, in feet.	Reading at Gauge No. 3,* in feet.	Percentage of submergence.*
1	3	0.525	0.420	80
	6	0.954	0.606	63
	12	1.548	1.380	89
	13	1.690	1.475	87
	14	1.900	1.63	86
2	1	0.672	0.45	67
	2	0.923	0.71	77
	3	1.116	0.83	74
	4	1.408	1.093	78
	5	1.979	1.66	84
3	1	0.782	0.655	84
	2	1.160	0.830	72
	4	1.745	1.30	75
4	1	0.688	0.418	60
	2	1.102	0.820	74
8	5	1.710	1.28	75
	6	2.156	1.72	80
10	8	2.156	1.75	81
Average.....	..	.....	.....	77

*At instant of flooding.

Conclusions.—It is believed that the data submitted in this discussion justify the following tentative conclusions:

1.—A Venturi flume, properly set, can be made to act as a control and indicate flow from a single gauge reading in accordance with a fully rational law.

2.—For all practical purposes, especially in irrigation work, the coefficient of discharge for such a structure may be taken as unity.

3.—For practical purposes, the throat length of a structure similar to that shown in Fig. 10 should not be less than the greatest depth expected at the gauge. For best results, a throat length equal to twice the depth is desirable.

4.—The necessary throat length may be somewhat reduced by carefully rounding the corners at the junction of the throat and the outlet.

5.—For carefully rounded approaches no entrance loss need be expected.

6.—Friction losses may be ignored unless the structure is long or extreme precision is required.

7.—The Venturi principle is not utilized and the device should not be called a Venturi flume. Control-section meter is suggested as a more appropriate name.

E. A. JACOB,* M. AM. SOC. C. E. (by letter).†—The Improved Venturi Flume is a type of measuring device that deserves to be used more generally on irrigation canals. However, there are practical limitations to its adoption.

Some time ago, the writer installed a test Venturi flume in a canal having a grade of 0.0003, and carrying about 60 cu. ft. per sec. of water. The canal was in clay which would withstand a velocity of 3 to 4 ft. per sec. The flume itself operated well, but the canal below it was inadequate for the velocity of the

* Engr., Provo Reservoir Co., Provo, Utah.

† Received by the Secretary, December 28, 1925.

outlet stream, about 5 ft. per sec. The flume remained in the canal a few days, and then was taken out.

Venturi flumes may be used only in canals where the material will not erode under the force of the current flowing from the flume; otherwise, the canal must be lined with concrete or other stable material for some distance below the flume. If this outflow velocity is kept low, that is, 3 to 4 ft. per sec., the throat of the flume must have a water cross-section relatively large as compared with that of the canal. This would produce a high percentage of submergence which would make the formula proposed by the author unreliable if not entirely erroneous.

In Utah, most of the canals have moss growths which, in July and August, greatly decrease their capacities and produce back-water effects that would make it necessary to take into account both the throat head, H_b and the upper head, H_a. It would seem that more experiments showing the effects of variable back-water conditions with a constant discharge should be made, and a formula developed for such conditions involving both heads.

J. C. STEVENS,* M. AM. SOC. C. E. (by letter).†—The Special Committee on Irrigation Hydraulics, of which the writer has the honor to serve as Secretary, feels that the importance of proper measurement of deliveries in irrigation work deserves especial emphasis. The pay-for-what-is-consumed principle, if generally adopted, will eliminate many of the perplexing problems of operators and managers. It will reduce waste, eliminate the "water hog", and conserve the available supply to such an extent that in many instances substantial increases in the irrigated acreage may be possible. The chief difficulty in the way of putting such a system into general practice has been the lack of practicable metering devices, such as the one described by Mr. Parshall.

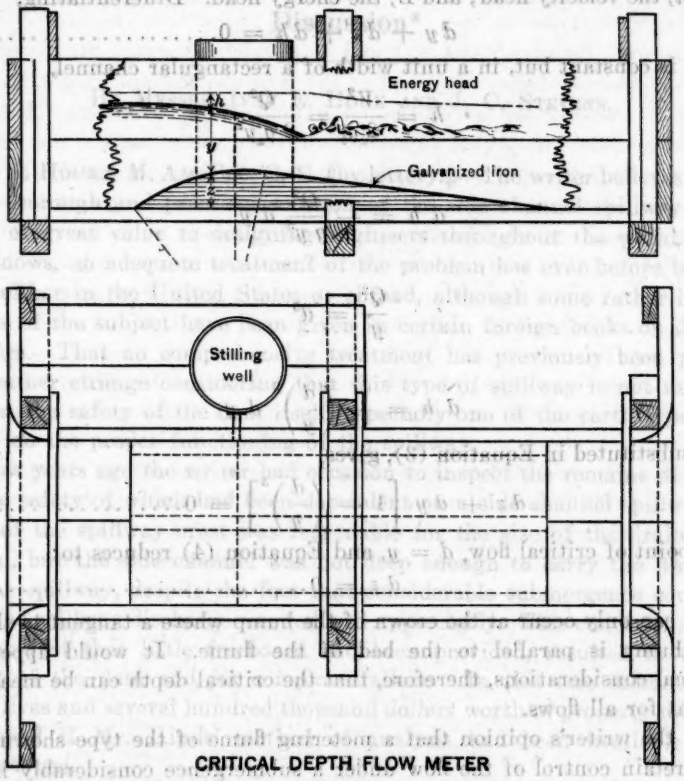
This is an excellent start in the right direction. It offers a means of metering the flow without sacrificing much head—a most desirable feature in flat countries. The writer, however, believes it can be improved and simplified to the point where the flow through it can be calculated from hydraulic principles.

Any device that requires the recording of two heads does not meet the requirements of simplicity. The Improved Venturi Flume requires only one observation of head to the point of about 70% submergence and, therefore, should be considered as a practical metering device up to but not beyond that limit.

The writer suggested to the Committee the use of a straight-sided, level-bottomed flume with a hump on the bed over which the water would have to flow at critical depth. In order to recover as much of the velocity head as possible the use of a second hump below the first was also considered, the purpose being to force the water from low stage below the critical to the high stage above without appreciable impact or eddy loss. A flume of this type was constructed and some experiments were recently made on it at the Fort Collins Hydraulic Laboratory by Mr. Parshall.

* Cons. Hydr. Engr. (Stevens & Koon), Portland, Ore.
† Received by the Secretary, December 28, 1925.

Although the results of these tests have not been fully analyzed it appears that the second hump is superfluous. Its function is to raise the water to the high stage without a hydraulic jump, but this can occur for only one rate of flow. The intention in design was that this rate should be the maximum and that for all lesser rates of flow a hydraulic jump would occur. The eddy loss in a low jump is practically negligible.* Thus, for a depth after the jump 25% greater than that before, the energy loss is only 0.17% and for a 50% increase in depth the loss is only 1.2 per cent.



CRITICAL DEPTH FLOW METER

FIG. 16.

Plans are being made, therefore, for laboratory tests on such a flume with a single hump in the bottom somewhat along the lines shown in Fig. 16. Julian Hinds, M. Am. Soc. C. E., suggested to the Committee the desirability of measuring the head at the critical depth. If this can be worked out in a practical way all question of velocity of approach, silting, weeds, etc., will be eliminated, and the formula for flow will reduce to the simple theoretical form:

$$Q = L \sqrt{g}\, d^{\frac{3}{2}} \quad \ldots\ldots (1)$$

in which, Q is the flow, in second-feet; L is the length of crest of the hump or the width of flume at the point of critical flow; and d is the critical depth.

* See "Determining the Energy Loss in the Hydraulic Jump," by J. C. Stevens, *Engineering News-Record*, June 4, 1925.

A coefficient may have to be introduced, since theory does not take account of variable velocities in the plane of critical flow.

Theoretically, the critical depth should remain at the same point on a hump of this kind for all rates of flow. Neglecting friction, the following relation is obvious at any point in the flume:

$$y + z + h = E \ldots\ldots (2)$$

in which, z is the height of the hump above the flume bed; y is the depth of water; h, the velocity head; and E, the energy head. Differentiating,

$$d\,y + d\,z + d\,h = 0 \ldots\ldots (3)$$

since E is constant but, in a unit width of a rectangular channel,

$$h = \frac{V^2}{2\,g} = \frac{Q^2}{2\,g\,y^2}$$

hence,

$$d\,h = -\frac{Q^2}{g\,y^3}\,d\,y$$

but,*

$$\frac{Q^2}{g} = d^3$$

hence,

$$d\,h = -\left(\frac{d}{y}\right)^3 d\,y$$

which, substituted in Equation (3), gives,

$$d\,z + d\,y\left[1 - \left(\frac{d}{y}\right)^3\right] = 0 \ldots\ldots (4)$$

At the point of critical flow, $d = y$, and Equation (4) reduces to:

$$d\,z = 0 \ldots\ldots (5)$$

This can only occur at the crown of the hump where a tangent to the curve of the hump is parallel to the bed of the flume. It would appear from theoretical considerations, therefore, that the critical depth can be measured at one point for all flows.

It is the writer's opinion that a metering flume of the type shown in Fig. 16 will retain control of the flow under a submergence considerably in excess of 70 per cent. The Committee awaits with considerable interest the results of the author's promised experiments with this type of measuring device.

The term, "Venturi Flume", is not a happy expression for the device described by the author, nor for the one suggested herein. The term, "Critical Depth Flow Meter", is more to the point and is offered as a substitute.

*At critical flow the velocity head is one-half the depth, that is, $\frac{V^2}{2g} = \frac{1}{2}\,d$, also, $V^2 = \frac{Q^2}{d^2}$; hence, $\frac{Q^2}{g} = d^3$.

SIDE CHANNEL SPILLWAYS: HYDRAULIC THEORY, ECONOMIC FACTORS, AND EXPERIMENTAL DETERMINATION OF LOSSES

Discussion*

By Messrs. Ivan E. Houk and J. C. Stevens.

Ivan E. Houk,† M. Am. Soc. C. E. (by letter).‡—The writer believes that Mr. Hinds' thorough and practical solution of the side channel spillway problem will be of great value to designing engineers throughout the world. So far as he knows, no adequate treatment of the problem has ever before been published, either in the United States or abroad, although some rather brief discussions of the subject have been given in certain foreign books on dams and reservoirs. That no comprehensive treatment has previously been published seems rather strange considering that this type of spillway is not uncommon and that the safety of the dam itself, especially one of the earth type, is contingent on the proper functioning of the spillway.

A few years ago the writer had occasion to inspect the remains of an earth dam the safety of which had been dependent on a side channel spillway. The length of the spillway crest was reasonable for the size of the drainage area involved, but the side channel was not deep enough to carry the water away from the spillway, despite the fact that considerable submergence could occur before the spillway discharge was reduced appreciably. This condition, together with the fact that little freeboard had been provided, resulted in the overtopping of the dam and its complete failure. In this one instance, alone, several lives and several hundred thousand dollars worth of property might have been saved if Mr. Hinds' method of analysis had been available for the designer's use.

The unusually high losses that occur in the side channel spillway are due to the impact of the spillway discharge with the flow in the channel, as Mr. Hinds explains. If the topographic conditions are such that the spillway discharge can be led directly away from the apron, into a gradually converging and curving channel, the direction of flow probably can be changed through an angle of 90°, from a direction normal to the spillway crest to a direction parallel with the spillway, with relatively small losses. The large impact losses will be avoided and probably some of the velocity head attained in falling over the spillway will be conserved. It is doubtful whether the loss caused by curvature in open channels ever amounts to 100% of the velocity head in the case of

* Discussion of the paper by Julian Hinds, M. Am. Soc. C. E., continued from January, 1926, *Proceedings*.

† Engr., U. S. Bureau of Reclamation, Denver, Colo.

‡ Received by the Secretary, December 22, 1925.

90° bends. At any rate, that was the conclusion reached by the writer some years ago, after making careful experimental determinations of losses in curves in open channels carrying discharges of from 200 to 70 000 sec-ft. at velocities of from 2 to 10 ft. per sec. The controlling factor in spillway location, however, is not the magnitude of the losses in the spillway channel, but the total cost of an adequate structure. In many instances the cost of a side channel spillway, with proper provision for the large losses involved, will be less than that of any other type.

The author's Figs. 16,* 17,† 18,‡ and 25,§ which illustrate the experimental work at the Bellvue Laboratory and at the Arrowrock Dam, might have been improved by showing the actual platted water surface elevations. The photographs in the paper show that in several runs the water surface was comparatively rough, and, therefore, the reader may wonder how accurately the surface curves were determined and how widely the individual observations varied from the smooth curves shown. The writer knows that the curves were accurately determined and that the variation of the measured points from the curves was not excessive, inasmuch as it was his privilege to assist Mr. Hinds in the Bellvue experiments. The matter is simply mentioned for the benefit of those who may be interested.

The Engineering Profession undoubtedly will accept Mr. Hinds' solution of the side channel spillway problem, as it stands. The theory is based on one of the best established laws of physics; the results of the laboratory experiments at Bellvue show substantial agreement with the theoretical formulas; and the results of the field tests on the full-sized structure at the Arrowrock Dam remove all doubt as to whether the theoretical formulas found to be applicable to small-sized laboratory models will apply to full-sized structures in actual practice.

J. C. STEVENS,‖ M. AM. SOC. C. E. (by letter).¶—This is a most excellent presentation of an intricate hydraulic problem. If the fundamental basis of the author's mathematical analysis be accepted, that the water overflowing the spillway crest has no kinetic energy in the direction of continued motion, then his conclusions must be accepted for they follow rigidly therefrom. The author states his hypothesis a little more generally in observing that when two streams of water at different velocities come together and flow away together the energy of impact is lost.

In order to test the soundness of this hypothesis the Special Committee on Irrigation Hydraulics, of which the writer has the honor to serve as Secretary, has arranged for certain experiments at the Hydraulic Laboratory at Berkeley, Calif., that are now (December, 1925) under way. These experiments are under the direction of B. A. Etcheverry, M. Am. Soc. C. E., and B. A. Vanleer, Assistant Professor of Mechanical Engineering, University of California, who

* *Proceedings*, Am. Soc. C. E., September, 1925, Papers and Discussions, p. 1378.
† *Loc. cit.*, p. 1379.
‡ *Loc. cit.*, p. 1380.
§ *Loc. cit.*, p. 1389.
‖ Cons. Hydr. Engr. (Stevens & Koon), Portland, Ore.
¶ Received by the Secretary, December 24, 1925.

is in immediate charge of the Laboratory. It is also hoped that other experiments with the same object in view will be conducted at the Hydraulic Laboratory of the Oregon Agricultural College at Corvallis, Ore., under the immediate direction of H. S. Rodgers, Professor of Hydraulics.

These experiments will endeavor to determine the head lost by impact and eddies when two streams of water intersect at right angles in pipes. Pipes are being used in preference to open conduits because greater heads can be secured and the water is free from entrained air.

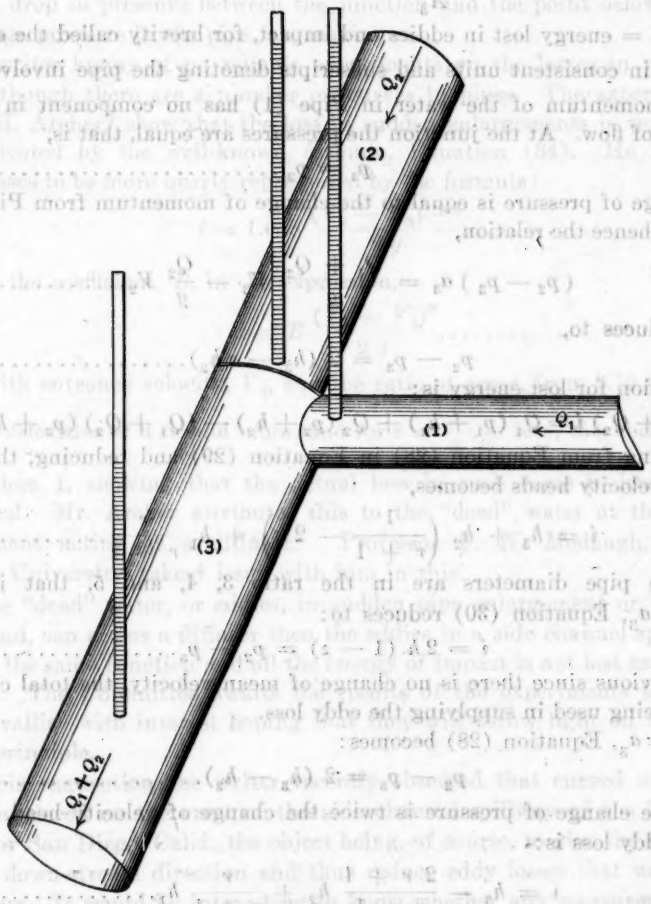


FIG. 31.

The writer submits herewith the mathematical treatment of the hydraulics of intersecting pipes, based on the theory in the paper. If the experiments prove the correctness of the formulas herein the theory may be accepted as sound.

Let two pipes flowing full under pressure intersect, as shown in Fig. 31.

Let d = inside diameter of the pipe;
a = area of pipe;
p = pressure head;
Q = flow;
v = mean velocity;
h = velocity head;
r = ratio of initial discharges $= \frac{Q_1}{Q_2}$;
z = ratio of areas, $\frac{A_2}{A_3}$; and
i = energy lost in eddies and impact, for brevity called the eddy loss;

all being in consistent units and subscripts denoting the pipe involved.

The momentum of the water in Pipe (1) has no component in the final direction of flow. At the junction the pressures are equal, that is,

$$p_1 = p_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

The change of pressure is equal to the change of momentum from Pipe (2) to Pipe (3), hence the relation,

$$(p_2 - p_3)\, a_3 = \frac{Q_1 + Q_2}{g}\, V_3 - \frac{Q_2}{g}\, V_2 \quad \ldots\ldots\ldots\ldots\ldots (27)$$

which reduces to,

$$p_2 - p_3 = 2\,(h_3 - z\,h_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

The equation for lost energy is:

$$(Q_1 + Q_2)\, i = Q_1\,(p_1 + h_1) + Q_2\,(p_2 + h_2) - (Q_1 + Q_2)\,(p_3 + h_3) \ldots (29)$$

Substituting from Equation (28) in Equation (29) and reducing, the loss in terms of velocity heads becomes,

$$i = h_3 + h_2 \left(\frac{1}{r+1} - 2\,z\right) + h_1 \frac{r}{r+1} \quad \ldots\ldots\ldots\ldots (30)$$

When the pipe diameters are in the ratio 3, 4, and 5, that is, when, $a_1 + a_2 = a_3$, Equation (30) reduces to:

$$i = 2\,h\,(1 - z) = p_2 - p_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

This is obvious since there is no change of mean velocity, the total change of pressure being used in supplying the eddy loss.

If $a_2 = a_3$, Equation (28) becomes:

$$p_2 - p_3 = 2\,(h_3 - h_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (32)$$

that is, the change of pressure is twice the change of velocity head. In this case the eddy loss is:

$$i = h_3 - \frac{2\,r+1}{r+1}\, h_2 + \frac{r}{r+1}\, h_1 \quad \ldots\ldots\ldots\ldots\ldots (33)$$

When Q_1 is zero, r becomes zero and Equation (30) reduces to:

$$i = \frac{(V_2 - V_3)^2}{2\,g} \quad \ldots\ldots\ldots\ldots\ldots\ldots (34)$$

which is the eddy loss in a sudden enlargement.

If Q_2 is zero, Equation (27) reduces to:

$$p_1 - p_3 = 2\,h_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (35)$$

and the eddy loss is:

$$i = h_1 + h_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36)$$

which gives the loss in a right-angle bend. If Pipes (1) and (3) are of the same size the velocity is constant and the loss is twice the velocity head, or,

$$i = 2\,h = p_1 - p_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37)$$

This is comparable with Equation (31) under conditions of no change in velocity, the entire drop of pressure going to supply the eddy loss.

These formulas do not take account of friction losses. In order to isolate the eddy losses the friction losses must be calculated separately and deducted from the drop in pressure between the junction and the point below it where the average pressure in the pipe is a maximum.

The writer knows of no reliable experiments on the losses in right-angle bends, although there are a number on curves in pipes. The experiments by Mr. W. H. Archer* show that the loss in sudden enlargements is very close to that indicated by the well-known formula, Equation (34). He found the actual losses to be more nearly represented by the formula:

$$i = 1.098\,\frac{(V_2 - V_3)^{1.919}}{2\,g} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38)$$

and that the coefficient, B, in the expression,

$$i = B\,\frac{(V_2 - V_3)^2}{2\,g} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39)$$

varied with entrance velocity, V_2, and the ratio of areas, from 0.75 to 1.2; and that, for velocities of 6 ft. and more, and for $z = \frac{1}{2}$, or less, the coefficient, B, is less than 1, showing that the actual loss in such cases is less than the theoretical. Mr. Archer attributes this to the "dead" water at the point of enlargement acting as a diffuser. Professor F. W. Medaugh, of Johns Hopkins University, takes† issue with him in this.

If the "dead" water, or eddies, in sudden pipe enlargement or in a right-angle bend, can act as a diffuser then the eddies in a side channel spillway can perform the same function and all the energy of impact is not lost as the author assumes. The Committee awaits the results of the experiments at Berkeley and Corvallis with interest hoping that they will throw light on this fundamental principle.

In this connection the writer recently observed that curved metal vanes had been placed on the crest of the side channel spillway of the Sweetwater Dam near San Diego, Calif., the object being, of course, to give the overflowing water a down-stream direction and thus reduce eddy losses that would otherwise occur. It would be interesting to know whether any measurements have been made of the efficacy of these vanes.

Mention may be made of the tests on models of the side channel spillways of the Burrinjuck Dam‡ on Murrumbidgee River in New South Wales. These

* *Transactions*, Am. Soc. C. E., Vol. LXXVI, (1913), p. 999.

† *Engineering and Contracting*, January 10, 1923.

‡ "The Spillways of Burrinjuck Dam," by Henry Harvey Dare, *Minutes of Proceedings*, Inst. C. E., Vol. CCXIV, Session 1921-22, Part II.

were originally designed to pass a flood of 40 000 sec-ft. While the dam was under construction in 1916 a flood of about 65 000 sec-ft. passed over the works. Because of this, models were built and tests made to ascertain if increased capacity could be secured within the limits of practicable construction. As a result the south spillway was lengthened and then both spillways were deepened more than the original plans indicated, to take a flood corresponding to that of 1916 under a head of 10 ft. on the two spillways. In May, 1924, the greatest flood of record occurred. The water rose until the head on the spillways was 19.3 ft., the parapets of the dam being overtopped 3.3 ft.* The paper does not contain sufficient data to enable comparisons to be made of the model tests and the author's analysis.

The Committee hopes to secure in due time actual tests on the side channel spillway of the Rimrock Dam on Tieton River, Washington, recently completed by the U. S. Bureau of Reclamation, and also on that of the Lake Fordyce Spillway under construction by the Pacific Gas and Electric Company of California.

* *The Illustrated London News* of July 18, 1925, contains photographs of this flood.

THE ORADELL DAM OF THE HACKENSACK WATER COMPANY

Discussion*

By Messrs. Thaddeus Merriman, Lazarus White, Albert C. Arend, and Paul E. Gisiger.

Thaddeus Merriman,† M. Am. Soc. C. E.—Mr. Critchlow‡ touched on a very important point when he spoke of the spillway capacity. The maximum recorded floods as reported by the author are about 70 cu. ft. per sec. per sq. mile. This figure, however, does not represent the possibilities. Since the design of the Oradell Dam was made there has been recorded§ a flood discharge of 171 cu. ft. per sec. per sq. mile from a drainage area of 192 sq. miles on Esopus Creek in the Southern Catskills, New York State. Had the storm which produced this flood been centered over that water-shed, the resulting discharge would have been substantially greater.

Among the great storms on record is that of October, 1869,‖ which was central near Canton, Conn., where the total rainfall was 12.35 in. At Middletown, Conn., out of a total precipitation of 9.37 in., 7.15 in. fell in 24 hours. This quantity of rainfall in 24 hours compares with 10.42 in. as measured¶ at Newark, N. J., on October 8 and 9, 1903.

The design of the Oradell Dam is most interesting and instructive, because, at first glance, it seems to be turned 180° from a normal position. Close study, however, reveals a most carefully executed design, admirably adapted to meet the local conditions.

The design of any structure constitutes about one-half the entire story; the other half is in the kind and quality of the materials of which the structure is put together. This dam was built of concrete and it would be of value if, for the purpose of record, the author would describe the sands and the aggregates that were used. A statement as to the source of these materials, as to the kind of cement used, and as to the methods of mixing and proportioning, with particular reference to the water-cement ratio, would be most helpful to future students of concrete.

With regard to most structures the history of the materials entering into them is unknown and it is therefore practically impossible to trace back and

* Discussion of the paper by Nicholas S. Hill, Jr., M. Am. Soc. C. E., continued from January, 1926, *Proceedings*.

† Chf. Engr., Board of Water Supply, City of New York, New York, N. Y.

‡ *Proceedings*, Am. Soc. C. E., January, 1926, Papers and Discussions, p. 119.

§ *Loc. cit.*, May, 1925, Papers and Discussions, p. 906.

‖ *Transactions*, Am. Soc. C. E., 1878 (VII), p. 224.

¶ *Water Supply and Irrigation Paper No. 92*, U. S. Geological Survey.

determine the probable causes of disintegration or failure if it occurs. Records of this kind regarding many structures of the past twenty years would, to-day, be of great value.

LAZARUS WHITE,* M. AM. SOC. C. E.—The writer wishes to call attention to the dangers of a dam founded on wooden piles. In case of settlement of the material below the dam a space will be created through which circulating water may wash away such material with the danger of "piping". Of course the steel sheet-piling cut-off is designed to prevent this, but the danger is there in case of any defect of the cut-off.

The wooden piles were driven to sustain safe loads of 20 tons as determined by the *Engineering News* formula. The speaker's experience has demonstrated to his satisfaction the utter unreliability of the so-called *Engineering News* formula when checked with tests made by reliable hydraulic apparatus. Within a few weeks piles driven to "refusal" in a stiff clay not very far from the Oradell Dam could only sustain a load of a few tons from a hydraulic jack. He does not believe in the variation of the *Engineering News* formula proposed as "a formula applicable to all methods of pile-driving, including drop-hammer, single-acting steam hammer and double-acting steam hammer." In his opinion, the only safe way to determine the value of piles is to test a group of them against an actual load. Where piles are closely spaced the test of a single one may be deceptive because of overlapping effects.

As to the rusting of steel sheeting when buried below the ground, the speaker has often noted that it lasts very well. Steel sheeting dug up after being buried for about ten years was surrounded by a coating of rust which, when removed, disclosed that the sheeting itself retained nearly its full cross-section. This is also true of steel pipe buried in the ground. The coating of iron rust which forms in contact with the steel preserves it from the attack of oxygen in the ground or in circulating water.

ALBERT C. AREND,† M. AM. SOC. C. E. (by letter).‡—The writer believes that the conditions found to exist at the Oradell Dam are very similar to those found at points on the Big Blue and Republican Rivers, in Nebraska, but where no funds have been available to provide for such costly construction. These structures were evidently safe during all stages of construction work, and consequently they must have been unnecessarily safe after the earth-fill was properly completed.

The author's list of ten special conditions§ does not include any specific cost limitation although Provision No. 6 indicates that there was a limit.

The life of the completed work will evidently be measured by that of the steel sheet-piling. The end of the east abutment, where the steel sheet-pile cut-off stops, is evidently the weakest point of the structure, and the next weakest point would probably be where this solid masonry wall ends in the clay and white sand.

* Pres., Spencer, White & Prentis, Inc., New York, N. Y.
† Cons. Engr., Omaha, Nebr.
‡ Received by the Secretary, December 8, 1925.
§ *Proceedings*, Am. Soc. C. E., October, 1925, Papers and Discussions, p. 1569.

This criticism merely calls attention to the comparative inconsistency of a very safe structural design having ends or connections of only one-half or one-fourth the resistance of the connecting or adjacent parts. "A chain is still as strong as its weakest link."

It would be interesting to know the cost of this dam, and what the cost studies showed in comparison with a masonry section of gravity type extending to hard foundation. Then, if not already investigated, comparison could be made of the cost of a Taintor gate section of equal capacity and of an earth dam with sheet-pile cut-off.

Where the density of population to be served (or menaced) and the volume of business warrant the cost of an abnormally safe type of design, that part of the cost which might be termed security insurance is undoubtedly justified when it does not become a financial hindrance to the project; but how much less security, because of cost, would be justified where the population requirements, the volume of use, and the returns on the investment would not provide for any more than a necessary minimum cost? That is often the controlling and most important business feature connected with such work, and the writer would appreciate some discussion on this particular phase of the design.

PAUL E. GISIGER,* ASSOC. M. AM. SOC. C. E. (by letter).†—The dam described is of a very unusual and interesting type and its designers and builders are to be congratulated for discarding precedent and originating something new.

Studying the cross-section of the Oradell Dam, the question arises as to whether it would not be of advantage to go one step further, discarding the earth back-fill on the base slab and letting the water itself be part of the weight of the structure. Such a condition has in fact been assumed in the design, since it could have occurred while the dam was under construction.

It is easy to show that a dam of this type, the reverse of the conventional reinforced concrete structure, can be designed for the same degree of safety with a considerably smaller cross-section; and it is hard to understand why this never has been done before. An aversion to putting tension in the buttresses and in their connection to the face slab may partly account for this reluctance, but it did not stop engineers from building reinforced concrete retaining walls of such a type. A very good solution of this detail problem has been found in the Oradell Dam by the use of structural steel.

The writer has analyzed three skeleton designs of reinforced concrete dams, assuming similar height and foundation conditions. Fig. 13 (*a*) shows a dam of conventional design, Fig. 13 (*b*) the same cross-section with water acting from the other side, and Fig. 13 (*c*) a reduced cross-section, which is even safer than Fig. 13 (*a*). The accompanying computations show that for two dams of the same cross-sectional area, Types (*a*) and (*b*), the one with water standing on the base slab, Type (*b*), is considerably safer against sliding and overturning; but, for the same safety as with the conventional type,

* Care, Pennsylvania Water & Power Co., Baltimore, Md.
† Received by the Secretary, December 3, 1925.

Fig. 13 (*a*), the cross-section of the other type, Fig. 13 (*c*), can be reduced by at least 25 per cent.

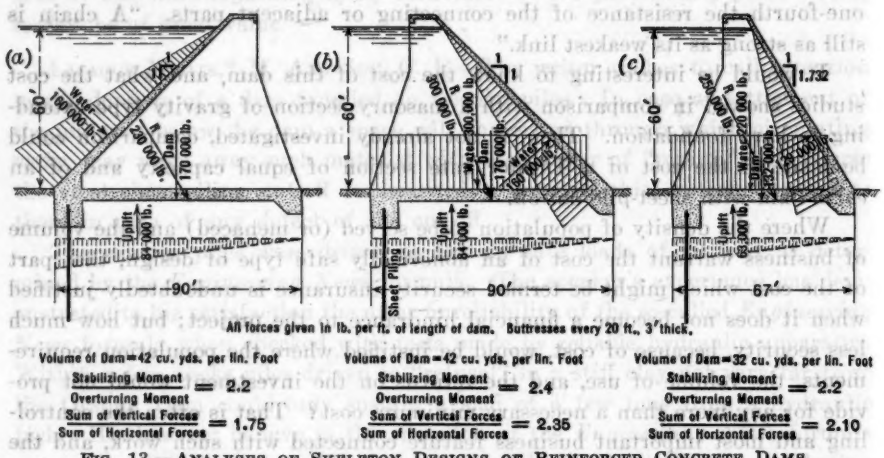

(a)
(b)
(c)
60'
Water 180 000 lb.
R 220 000 lb.
Dam 170 000 lb.
Uplift 84 000 lb.
90'
R 300 000 lb.
Water 200 000 lb.
Dam 170 000 lb.
Uplift 84 000 lb.
Sheet Piling
90'
1
1.732
R 250 000 lb.
Water 220 000 lb.
Dam 170 000 lb.
Uplift 63 000 lb.
67'
All forces given in lb. per ft. of length of dam. Buttresses every 20 ft., 3' thick.
Volume of Dam = 42 cu. yds. per lin. Foot
Stabilizing Moment / Overturning Moment = 2.2
Sum of Vertical Forces / Sum of Horizontal Forces = 1.75
Volume of Dam = 42 cu. yds. per lin. Foot
Stabilizing Moment / Overturning Moment = 2.4
Sum of Vertical Forces / Sum of Horizontal Forces = 2.35
Volume of Dam = 32 cu. yds. per lin. Foot
Stabilizing Moment / Overturning Moment = 2.2
Sum of Vertical Forces / Sum of Horizontal Forces = 2.10

FIG. 13.—ANALYSES OF SKELETON DESIGNS OF REINFORCED CONCRETE DAMS.

A further advantage that should not be underestimated is the easy adaptation of this cross-section to a spillway, whereas with the usual type the transition from dam to spillway as a rule presents many difficulties.

The objection might be raised that, due to the weight of water on the base slab, the foundation pressures become higher. This would be erroneous, however, since the load, instead of being concentrated under the buttresses as in the conventional type, will be almost evenly distributed over the whole base.

MOMENTS IN RESTRAINED AND CONTINUOUS BEAMS BY THE METHOD OF CONJUGATE POINTS

Discussion*

By Messrs. Hardy Cross, George Higgins, E. G. Walker, and Donald Derickson.

Hardy Cross,† M. Am. Soc. C. E. (by letter).‡—This paper is an interesting contribution to American literature in this field. It is especially welcome to the writer because the authors emphasize that it really treats of geometry—the geometry of ideal distortions in structures. Since the relations are geometrical, they can be presented in almost unlimited variety either in purely geometrical form (graphically) or in algebraic form (analytically). The choice between these two methods is almost entirely one of individual taste and experience. The writer prefers analytical solutions for all the problems discussed and finds them more expeditious, but this does not mean that he discounts at all the value of the graphical constructions presented.

The paper is free from what seems to be a fault of frequent occurrence in the literature of indeterminate structures, exaggeration of the value and importance of exact influence lines. With uniform loads or where equivalent uniform loads (the equivalence being strictly correct only for a triangular influence line) are available or easily determined, influence lines are useful chiefly in determining load divides. In continuous-girder problems one may determine by inspection what spans should be loaded, and uniform or equivalent uniform loads are sufficiently exact. It is only for points between the fixed points and the supports that divides for moment need to be found. In most cases, it is better to compute maximum moments directly rather than from influence lines.

The treatment of secondary stresses as due to bending moments in continuous girders on settling supports, either neglecting the web members or assuming pin connections at their far ends, is very interesting. The writer has at times indicated this method to his students, but without much faith in its general accuracy in the cases of secondary stresses which are worth computation. The truss used as an illustration by the authors is not such a case, and the writer hopes that they have available and will present further data on the degree of approximation of this method. It is probably not very close in such cases as arches, where the web is very stiff near the center.

* Discussion of the paper by L. H. Nishkian and D. B. Steinman, Members, Am. Soc. C. E., continued from January, 1926, *Proceedings*.

† Prof., Structural Eng., Univ. of Illinois, Urbana, Ill.

‡ Received by the Secretary, December 3, 1925.

The paper suggests many lines of study and comparison which space forbids developing. The writer has not investigated in detail the originality of the constructions as compared with those of Fidler or with later similar constructions which he is told are due to Ostenfeld. Indeed he does not consider the question very important. A contribution to American literature which is not merely uninterpreted translation—or, worst of all, undigested translation—has value in itself irrespective of comparisons with foreign literature; the literature of indeterminate structures has often been burdened by historical precedents which have limited it to so-called "classic", and very unusable, forms.

GEORGE HIGGINS,* M. AM. SOC. C. E. (by letter).†—This paper is a valuable addition to engineering literature on the subject of continuous beams and, as the authors suggest, will prove helpful to designers of structures in reinforced concrete. Not only is the authors' method an excellent one, but their explanation is remarkably clear and their industry in applying the method of conjugate points to the solution of many different problems will earn the gratitude of many engineers.

In 1908 and 1909, the writer gave considerable study‡ to the problem of determining the best spacing of the posts, corresponding to different kinds of loading on 2-post trussed beams. Had he used the method of conjugate points, the simplification probably would have been considerable. The trussed beam problem differs from those given as examples by the authors§ in that the ends are not fixed, but simply held down by the tie-rods when they tend to rise, and held up by the supports in all cases. In other words, negative bending moments occur. The writer has investigated several cases of partial uniform loading, and would be glad to communicate the results to other members.

E. G. WALKER,‖ M. AM. SOC. C. E. (by letter).¶—The writer has read this paper with considerable interest. He believes it to mark a very definite advance in the practical design of continuous beams and girders. Although the Theorem of Three Moments and its more advanced generalizations have long been well known to mathematicians and engineers, practical application has always been on a restricted scale by reason of the great amount of arithmetical computation involved in applying these principles to practical design.

The graphical methods developed by the authors should go far toward increasing the use of more exact methods of calculation for constructions in which continuity of structure over points of support is involved. It may be legitimate to apply methods which require much computation in proportion to the results obtained to large and heavy structures involving a considerable expenditure (for example, long-span swing bridges, continuous girder bridges, etc.), but it is not economical to incur the cost involved in elaborate calcula-

* Cons. Engr., South Yarra, Victoria, Australia.

† Received by the Secretary, December 14, 1925.

‡ "Trussed Beams," *Transactions*, Australasian Assoc. for the Advancement of Science, Vol. 83.

§ *Proceedings*, Am. Soc. C. E., October, 1925, Papers and Discussions, Fig. 20, Cases 10-13, p. 1611.

‖ London, England.

¶ Received by the Secretary, December 15, 1925.

tion to smaller structures, such as the framing of an ordinary reinforced concrete building.

The large advance which the paper makes toward enabling exact methods of computation to be applied in a relatively simple and easy manner, gives a great practical value to this Method of Conjugate Points. The writer regrets that he has not yet had an opportunity of using it in practice, but the principal conclusion which he has drawn from reading the paper is that the method is much more adaptable for ordinary office use than any other way of applying the Theorem of Three Moments to reinforced concrete design. It cannot compete in saving of time with the methods in general use for ordinary cases, in which empirical coefficients are used to calculate bending moments, but in structures of any appreciable magnitude the extra cost of the time involved in calculating by the authors' method should not be more than can be justified by increased accuracy of the results.

A great deal of the objection that has been taken in the past to the practical application of theories of continuity of structures has been based more on their complexities and lack of adaptability to the conditions of practical use rather than to valid opposition to the legitimacy of the basic assumptions used in developing the theories. Opponents of the application of exact statical methods have usually raised the objection that, in applying theories of continuity, conditions are assumed that are incorrect. In particular, it is frequently asserted that stiff connections, such as the intersections of beams and columns in a reinforced concrete building, are not sufficiently rigid to justify the assumptions which are made in regard to transmission of stress through them. On this basis it is assumed apparently that it is necessary for such intersections to be infinitely rigid in order to make calculations based on statical principles nearer the truth than those made by imposing an empirical multiplier on the moments calculated on an admittedly imperfect basis. While he does not subscribe to this opinion, the writer, probably in accordance with a great many other engineers of similar views, frequently has used approximate methods rather than consume the time necessary for more exact computation, because in many instances the saving which might be effected would not be sufficient to justify the cost of time involved in the longer calculations. Methods which are justifiable on a long continuous-span bridge are quite the reverse when applied to continuous, 15-ft. span floor-beams. It is in the direction of making closer calculations worth while in smaller structures that the value of the paper lies.

The authors have presented in an exceptionally clear manner the principles of this new method and have indicated very fully how it is applied to practical cases. The application is certainly simpler than that of the older Theory of Fixed Points and brings the whole subject within the domain of practical application to every-day work. A great advantage which should not be overlooked is the fact that with relatively little additional computation exact solutions are possible for cases of complex distributions of load as well as for variations of the moments of inertia.

Even with the simplification which the authors have effected it would hardly appear practical, however, in the ordinary way to apply such methods

as are developed in Table 1* to beams of the magnitude of the example which they have chosen. The writer, however, feels at a disadvantage in making much criticism on this point, because not having actually applied the methods of the paper to practical cases he is not qualified to judge how much of this computation can be saved by the use of the constants which the authors give in Table 2.† The computations of Table 1, however, are not more extensive than are necessary in the case of an ordinary arched rib. No doubt there are many cases which occur, as there are instances in the writer's own experience, where such amount of calculation is quite justifiable even on relatively small structural members.

In conclusion, the writer hopes that it will be possible to develop the principles outlined in the paper so that less reliance may be placed on the use of empirical figures and considerably more use made of the Theorem of Three Moments in the design of continuous structures in reinforced concrete and steel, thus enabling fuller advantage to be taken of the strength characteristics of the materials.

DONALD DERICKSON,‡ M. AM. SOC. C. E. (by letter).§—This interesting paper is a notable contribution to the subject of structural mechanics. It offers the most expeditious method of which the writer is aware for the solution of a certain type of problem which arises in structural engineering and especially pertains to reinforced concrete design. It has been his experience that too often methods have been used in determining the bending moments in reinforced concrete beams, which do not even approximately relate to actual conditions, and the convenient coefficients of the building codes and reports are applied to spans and loadings for which they were never intended. This may be ascribed to the lack of time necessary for the proper computations, or to ignorance of the principles involved, either or both of which have no doubt wasted many a dollar and probably will continue to do so. This paper presents methods, the applications of which are so easily made as compared with some of the better known ways of computation that it should have a beneficial effect on practice.

The writer appreciates the paper and like Oliver Twist would ask for more. In Section 9,‖ "Application to Indeterminate Frames," it is stated:

"If the structure or loading is unsymmetrical, a small secondary correction must be introduced on account of the slight lateral displacement of the middle span; this displacement is equivalent to a yielding of the respective supports of the outer span."

An application of the principles involved in Section 17,¶ to the solution of a problem based on assumed data would be of material assistance.

Can the methods of the paper be applied to a beam or girder in a building frame shown in Fig. 45?

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* *Proceedings*, Am. Soc. C. E., October, 1925, Papers and Discussions, p. 1617.
† *Loc. cit.*, p. 1618.
‡ Prof. of Civ. Eng., Tulane Univ., New Orleans, La.
§ Received by the Secretary, December 24, 1925.
‖ *Proceedings*, Am. Soc. C. E., October, 1925, Papers and Discussions, p. 1605.
¶ *Loc. cit.*, p. 1623.

The ends of the columns, h_2 and h_3, are neither free nor fixed and the same applies to the ends of the beams, l_1 and l_3; accordingly, it would seem that there would be some uncertainty as to the values to be used for the lengths of the respective members in computing the nominal length of the imaginary beam which is to be substituted for the actual members located at the ends of the beam under consideration. It would also be interesting to know what allowance should be made for the effect of the floor-slab in reinforced concrete construction, that is, what flange width should be used.

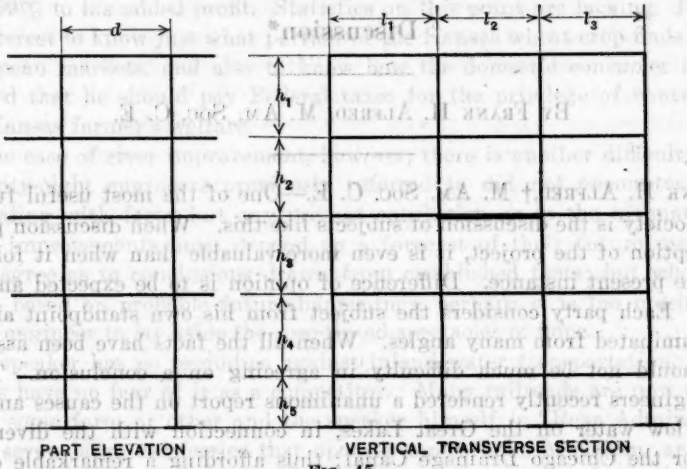

FIG. 45.

In the application of the principles of Section 17, to a reinforced concrete structure it would be of value to know what the authors would recommend for the Modulus of Elasticity, or if such a value may not be stated directly, how may an equivalent value be found?

Quoting from a widely used textbook regarding the moments in continuous beams, "the exact determination of the maximum bending moments at the critical sections of a beam of several spans is a long and tedious operation." This strengthens the student in his disinclination to give the subject as deep consideration as it merits. The increasing tendency toward introducing business courses into the curricula of engineering colleges is another factor that militates against the allowance of reasonable time for the study of continuous beams and indeterminate frames. Accordingly, it is most gratifying to have this paper available for use in class work.

RELATION OF THE OHIO RIVER AND ITS TRIBUTARIES TO TRANSPORTATION IN THE UNITED STATES

A SYMPOSIUM

Discussion*

By Frank H. Alfred, M. Am. Soc. C. E.

Frank H. Alfred,† M. Am. Soc. C. E.—One of the most useful functions of the Society is the discussion of subjects like this. When discussion precedes the inception of the project, it is even more valuable than when it follows it, as in the present instance. Difference of opinion is to be expected and is desirable. Each party considers the subject from his own standpoint and it is thus illuminated from many angles. When all the facts have been assembled, there should not be much difficulty in agreeing on a conclusion. Twenty-eight engineers recently rendered a unanimous report on the causes and remedies of low water on the Great Lakes, in connection with the diversion of water for the Chicago Drainage Canal; thus affording a remarkable example of the openness of mind with which the engineer should approach his problems.

When internal waterways is the topic, however, especial care should be taken to distinguish facts from fancies; for this subject has been so obscured in the popular mind by a sort of sentimental haze that it is easy to mistake one for the other and to accept bald assertion as Gospel truth. It is astonishing to find this lack of discernment in quarters where it would be least expected. For example, General Taylor‡ quotes from a Kansas enthusiast, presumably with approval. This quotation is taken from a pamphlet "published by the National Rivers and Harbors Congress for distribution among the agriculturists." It is a fair sample of propaganda on this subject and is also an excellent illustration of appeal to a special interest without considering the cost to others. This quotation in part is as follows:

"Since the farmer, in effect, pays the freight on all of his wheat to Liverpool (since that market fixes the price of all the wheat whether the wheat is actually exported or not), when we compute the value of Missouri River navigation to the Kansas wheat farmer, we cannot merely calculate the saving on the entire Kansas crop, because, if you raise the price of the export wheat 3 cents per bushel by reducing the cost of the freight to Liverpool by 3 cents, you automatically raise the price of all the wheat crop in Kansas."

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* Discussion continued from January, 1926, *Proceedings*.

† Pres. and Gen. Mgr., Pere Marquette Ry., Detroit, Mich.

‡ *Proceedings*, Am. Soc. C. E., December, 1925, Papers and Discussions, p. 2018.

This is followed by a computation which purports to show that the farmers of Kansas have lost $40 000 000 in 14 years on account of the lack of improvement on the Missouri River. This is based on an estimated saving in cost of transportation of 3 cents per bushel, a figure obtained "on good authority, in fact from experts." Assuming the existence of an expert wise enough to know what reduction in the freight rate would accrue, who pays the extra 3 cents to the Kansas farmer? Evidently, the person who buys his wheat. If 10% of this wheat is exported, 90% of the fortunate Kansan's increased emolument is contributed by his fellow citizens. If only 1% goes to Liverpool, they contribute 99% to his added profit. Statistics on this point are lacking. It would be of interest to know just what portion of the Kansas wheat crop finds its way to European markets, and also to know how the domestic consumer is to be convinced that he should pay Federal taxes for the privilege of contributing to the Kansas farmer's welfare.

In the case of river improvement, however, there is another difficulty which the twenty-eight engineers previously referred to did not encounter. They were dealing with facts, but any present conclusion as to the ultimate value of river improvements must depend on a forecast of their future use. It is easy to agree as to conclusions drawn from established facts; but where they must be based on probable future happenings, perhaps it is too much to ask even an engineer to lay aside the rose-tinted spectacles of hope.

The speaker has no prejudice against inland water transportation, and the railroads have no fear of it as a competitor. Many railroads are now engaged in it in some form or other and the speaker himself is "Rear Admiral" of a fleet of seven large car-ferries that operate across Lake Michigan, and are a very important part of the Pere Marquette's transportation system. Incidentally, he knows that the cost of this lake transportation per ton-mile of freight is as high as by rail, in spite of the fact that no maintenance-of-way expense is incurred. He maintains, however, that a broad distinction should be made between transportation on a great natural channel like the Great Lakes, or a great artificial channel like the Panama Canal where tolls are collected, and transportation on rivers, where immense sums are necessary to construct and maintain channels required even for boats of light draft.

Based on an experience of many years in the transportation field, his conclusion is that the future holds little of promise for river transportation, for the simple reason that except for the limited amount of traffic which originates and terminates on or near the river banks, it costs more to move freight from shipper to consignee by river than it does by rail. Transportation by river also requires more time and is less dependable. It is for these reasons that the rivers lost their traffic to the railroads and have never regained it and, in the speaker's opinion, never will regain it, no matter how much is spent on improving their channels. Patrick Henry's dictum as to how to judge the future still holds, and there is certainly nothing in the past history of traffic on the Ohio River to justify optimism. The expected increase in traffic has always been just around the corner, where it bids fair to remain.

The railroad takes freight at the shipper's mine or factory and delivers it at the consignee's warehouse. This the vessel cannot do, unless both

shipper and consignee are located at the water's edge. No comparison of transportation cost by rail and river is possible unless the terminal cost of delivery is included in the latter. The railroad rates include this cost on all but a negligible part of goods handled, but neither this cost nor any profit to the carrier is included in the figures which have been given in this discussion as representing the cost of river transportation. To give an idea what terminal costs amount to, the Pere Marquette Railway Company in 1923 had nearly one-half as many switching miles with locomotives at terminals as it had road miles in hauling the goods.

As to the alleged necessity for improved waterways to care for the greatly increased traffic of the future, why not let future generations attend to this if and when the need develops? The Ohio River improvement when complete will have cost $114 000 per mile, enough either to build and equip a double-track railroad, or to add a second track to each of the three existing single-track lines, which would give each railroad a capacity equal to at least ten times the present tonnage carried on the river. At this rate it will be a long time before extensive river improvements are called for on this account. Give the railroads a fair chance—which means simply enough return to attract the capital necessary for additions and betterments—and they will render cheaper and better service even to river communities than can be given by the river, and there will be no more of the congestion to which reference has been made.

It is hardly fair to refer to congestion that occurred under war-time conditions, or in 1920 when the railroads had not recovered from Federal control. The railroads are progressing, just like everything else. The total tractive power of locomotives used in the United States increased from 1 932 000 000 lb. in 1914 to 2 673 000 000 lb. in 1924, an increase of 38% in 10 years. In the same period the number of freight cars increased 16% and the average capacity of the freight car increased 12%, resulting in a total increase of freight car capacity of 30 per cent. In addition, there has been a large increase in the average distance which each freight car moves per day. Traffic in 1923 was 414 000 000 000 ton-miles as against 288 000 000 000 in 1914, an increase of 44 per cent. Taking all factors into consideration the railroads are capable of giving at least 60% more service now than they were ten years ago. In the second week of January, 1925, they handled the largest tonnage of coal ever handled in the history of coal mining or railroad operation. They actually moved 12 500 000 tons of coal in this one week. They never felt it; in fact, they did not know until afterward that they had moved nearly twice as much coal as in the previous week.

Let waterways be improved where the economic value of such improvement can be demonstrated, but let not assumption be mistaken for proof. Further, let it be assured that the resulting benefit will be fairly well distributed among the general public, which pays the cost, and not accrue only to certain favored interests or localities.

UTILIZABLE CAPACITY OF STEEL MEMBERS OF STRUCTURES

Discussion*

By Messrs. C. A. P. Turner, H. F. Moore, G. B. Woodruff, and R. von Fabrice.

C. A. P. Turner,† M. Am. Soc. C. E. (by letter).‡—As a study to discover the fundamental facts regarding the capacity of structural steel members the paper is a failure. It is wanting in a concise statement of vital facts from which definite conclusions may be drawn and it is encumbered with terms of nebulous significance.

Attributing to auto-stresses both the defects of improvements and elasticity will prove more confusing to the average reader than blowing hot and cold with the same breath did to the satyr in the fable. It is recognized that stresses in general may be normal or tangential, positive or negative; one wonders in reading the paper to what gender or classification these auto-stresses may be assigned, what they are, how are they generated, and what is their reaction.

Under tension or compression the position of a piece of steel in the Antimony-Bismuth scale changes in proportion to the stress applied within the elastic range of stress, in one direction for tension and in the opposite direction for compression, so that it is possible to measure dead stress by thermo-electric means. When the elastic range has been exceeded, however, the residual effect as to position is a change characteristic of the opposite kind of external stress to that causing the permanent set. This residual effect has been termed a "permanent strain" by some writers, associating it with the permanent set which accompanies but is not proportional to it. By others this effect has been termed "residual strain" or a strain that is left after exceeding the true elastic limit of the piece and is in equilibrium with, or balanced by, elastic strain.

Having opposite physical characteristics it is natural to expect opposite reactions or changes of shape when the piece is cut so that the elastic and residual strain is thrown out of equilibrium. Take several bars $\frac{3}{8}$ in. square, bent cold in horseshoe form, and measure the out-to-out dimension. Then place one in a vise and cut saw kerfs $\frac{1}{8}$ in. deep at $\frac{1}{2}$-in. intervals on the outside of

* This discussion (of the paper by Henry S. Prichard, M. Am. Soc. C. E., published in November, 1925, *Proceedings*, and presented at the meeting of December 9, 1925), is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

† Cons. Engr., Minneapolis, Minn.

‡ Received by the Secretary, November 11, 1925.

the bend. Release from the vise and note the inward spring of the ends. Cut another in a similar manner on the inside and again measure the inward spring of the ends. Take a third sample, cut both sides, and measure the outward spring of both ends, then the reaction of elastic strain as opposed to the contrary reaction of the residual strains in the outer fibers in the first and second experiments. Again, take a steel bar as it has passed through the rolls and saw kerfs as before on either top or bottom and, in general, the bar will assume a slight curvature. Cut on the sides and no change results. In like manner, cutting a keyway in a piece of cold-rolled shafting will cause the piece to assume a change in alignment.

A state of permanent strain may be defined as a state of internal stress comprising elastic and residual stress in equilibrium, from which no change in shape results unless this equilibrium is disturbed by forces supplying energy from without. Energy to change the internal state of equilibrium may be introduced as follows: (1) By temperature; (2) by applied forces; and (3) by cutting away a part of the symmetrically strained body. Such removal or cutting away of symmetrical parts of an over-strained elastic material permits the permanent strains that have been thrown out of equilibrium to re-assert themselves and change the form of the body in a manner comparable to applied external forces.

Gradual recovery from residual strain is sometimes termed elastic after-working and the lag or slow increase of deformation under stress is called hysteresis. These phenomena are more readily observed by twisting a fine wire suspended from a fixed support and having a small mirror attached to the lower end, the position of which can be observed in the usual way by means of a telescope and scale. If the lower end of the wire is twisted through a moderate angle and then left to itself, the mirror makes oscillations the extent of which may be read on the scale. These oscillations diminish more rapidly than when the only retarding force is the resistance of the air, showing that the force of torsion in the wire must be greater when the twist is increased than when it is diminishing.

According to Maxwell:*

"If we begin by keeping the wire twisted (at not too great or too small an angle) say, for a minute or an hour, and then leave it to itself, we find that the point of temporary equilibrium is displaced in the direction of twisting, and that this displacement is greater the longer the wire has been kept twisted. But this displacement of the point of equilibrium is not of the nature of a permanent set, for the wire, if left to itself, creeps back towards its original position, but always slower and slower. This slow motion has been observed going on for more than a week, and also if the wire was set in vibration the motion of the point of equilibrium was more rapid than when the wire was not in vibration.

"We may produce a very complicated series of motions of the lower end of the wire by previously subjecting the wire to a series of twists. For instance, we may first twist it in the positive direction, and keep it twisted for a day, then in the negative direction for an hour, and then in the positive direction for a minute. When the wire is left to itself, the displacement, at first positive, becomes negative in a few seconds, and this negative displacement increases for some time. It then diminishes and the displacement becomes positive and lasts a longer time till it too finally dies away."

* Encyclopedia Britannica. "Constitution of Bodies."

The phenomena here described represent an angle of distortion greater than the true elastic limit but less than the yield point of the metal. Kelvin has treated these phenomena as the "viscosity" of elastic solids. Experiments on the torsional vibration of wires detailed by Kelvin,* indicate elastic fatigue for a considerable range of stress within the limit of elasticity of shape. The experiments were as follows:

About April 26, two equal and similar lead weights or vibrators were hung up by equal and similar copper wires, the upper and lower ends of the two wires being similarly fixed by soldering. Wire No. 2 was kept vibrating as continuously as possible from day to day, and the other kept at rest except when vibrated once a day in an experiment to test the comparative viscosity of the wires, but no exact comparison was made until May 15. Then Wire No. 1 in 97 vibrations subsided from an initial range of 20 to a range of 10, while Wire No. 2 gave the same subsidence in 98 vibrations. During the greater part of May 16 and 17, Wire No. 2, was kept vibrating and Wire No. 1 quiet, and experiments were made with the result that Wire No. 1 subsided from an initial range of 20 to 10 in 98 to 99 vibrations of 2.4 sec. each, while Wire No. 2 gave the same subsidence in from 57 to 60 vibrations of 2.4 sec. each. It was supposed from these experiments that true elastic fatigue occurred for all stresses within the limit of elasticity of shape until a similar and more extensive series of experiments conducted by H. Tomlinson† showed that no such fatigue of elasticity is felt in steel, iron, or copper, provided the stress does not exceed a certain well-defined limit.

The "micro-flaw" theory of fatigue advanced by Johnson‡ in untenable because of its failure to account for observed phenomena. Thus, the elastic fatigue in Thompson's experiments depending on the immediately previous molecular condition, whether of quiescence or recurring changes of shape, cannot be accounted for by micro-flaws which are necessarily permanent and not temporary. When a bar of steel is loaded a number of times alternately in tension and compression to a point a little beyond the yield points in tension and compression, these points are thereby both reduced to about six-tenths of their original values (called by Bauschinger the "natural elastic limits" of the material). When the material is repeatedly strained in tension from zero to a value above the natural elastic limit in tension, but within the yield point, the yield point value in tension is raised by such treatment and, in like manner, when similar stresses above these natural elastic limits, but within the yield point values, are applied alternately from tension to compression, the normal yield point in tension is lowered. These phenomena raise the question whether the temporary strain which accounts for the elastic after-working is of the same nature as that produced by a permanent set, and what the limits of its accumulation may be under repeated stress in working of the material, without giving it an opportunity to recover by elastic after-working between consecutive loadings.

* Encyclopedia Britannica, "Elasticity."
† *Philosophical Transactions*, Royal Soc., 1886.
‡ "Materials of Construction."

These questions would seemingly be more readily investigated by considering the thermal stress curves of the material under load. During elongation soft steel grows uniformly cooler until a stress of about five-eighths the yield point is reached, when the heat of imperfect elastic deformation causes the curve to depart from the straight line. This continues until the yield point is reached when there is a sudden evolution of heat. In like manner, under compression the straight-line relation of temperature increase is maintained until a stress of about five-eighths the yield point is reached, then a similar phenomenon occurs.

The identity of cause (that is, thermal effect), would indicate an identity of result. In other words, for the range of imperfect elasticity within the yield point under repeated working of the material beyond its natural elastic limits, the residual strain accumulates with each repetition of the applied load even if no apparent set or plastic deformation is noticeable. As the apparent yield point is gradually raised by such treatment, the range of stress may be gradually increased until the ultimate resistance of the material is reached without noticeable plastic deformation. The same piece of steel under a single application of the load would stretch 20% of its length before "necking down" and finally breaking.

The fact that a residual strain is of contrary sign to the internal stress opposing the force which produced it leads at once to the question whether a change in the yield point is a phenomenon of residual strain or of change in structure. Microscopic examination fails to reveal any change in the material that has undergone a sufficient number of repetitions above its natural elastic limits to nearly wear it out or cause it to approach the point of rupture.

The behavior of iron and steel under fluctuating or repeated stresses was the subject of long research by Wöhler and later by Bauschinger. The result of these experiments are claimed to show:

1.—That resistance to fracture under stresses fluctuating within certain limits depends on the difference between the maximum and minimum applied stresses rather than on the maximum stress alone.

2.—That reversed stresses (tensile and compressive), much below the static breaking stress, are sufficient to cause fracture if repeated a great number of times.

Experiments made by the Phœnix Company on axle iron having an ultimate strength in tension of 46 000 lb. with an elongation of about 20%, gave limiting maximum stresses under limited ranges as shown in Table 23.

TABLE 23.—Tests on Axle Iron by Repeated Stress.

Kind of repeated load.	Stress, in Thousands of Pounds per Square Inch.	
	Limiting maximum.	Limiting range.
Completely reversed	15.2	30.4
From maximum to zero	30.56	30.56
From maximum to half load	41.02	About 20

The number of repetitions of stress without breaking, from which these limiting ranges were determined, amounted to millions. The specimens unbroken at an alternating stress of 15 200 lb. per sq. in. were subjected to 132 000 000 repetitions without breaking. This limiting range, which allows nothing for dynamic effect, is nearly six-tenths of the normal yield point of the material; if the dynamic effect of the rapidity of repetition of stress amounts to 10%, the limiting range would be about two-thirds of the normal yield point.

The ultimate strength of the material was 46 000 lb. per sq. in. and its normal yield point was about 24 000 lb. per sq. in. The limiting maximum stress of 42 000 lb. per sq. in. for repetitions of half load in Table 23 is, therefore, at least 18 000 lb. more than the normal yield point, and since whenever the normal yield point is exceeded a large deformation results, the manner in which these repeated stresses were applied is of interest. To avoid the large deformation produced when the normal yield point is exceeded, the stresses first applied in these experiments were less than the normal yield point; this point was then gradually raised until its ultimate position was above the maximum stress considered in the range noted as the limiting range.

In a practical structure the conditions of loading are not such as to raise the yield point of the material and thus increase its resistance, as was done in these experiments, and failure would result from only a few repetitions beyond the yield point. Consequently, the limiting ranges of intensity of stress which the practical structure can withstand cannot be deduced from experimental results where such artificial assistance to the resistance of the material is given.

As advocates of the fatigue formulas of proportioning agree that, if it be established that Wöhler's results are accounted for by the effects of stress beyond the true limits of elasticity, they have no relation to the logical determination of working stresses, this matter will now be considered from the standpoint of the three determinants of ideal elasticity, to wit:

1.—Proportionality of temperature change to the load applied;

2.—Proportionality of elastic elongation to the applied stress or load; and

3.—The reversible cycle of mechanical work during loading and unloading.

No bar of sound metal has ever been broken by Wöhler's treatment under ranges of stress of one kind or of both kinds, that is, from tension to compression within the limits of thermal proportionality. *Per contra* the range of stress from tension to compression within the natural elastic limits was found by Bauschinger to improve the metal in elasticity, tenacity, and ductility instead of injuring it.

The radical improvement in the strength of cast test bars and the suggestion that they shall not be "rumbled"* (treated in a tumbling barrel for cleaning) if their modulus of rupture be taken to represent the average condition of the casting, is a striking confirmation of the beneficial effect of stress ranging from tension to compression in reducing and eliminating residual states of stress arising from shrinkage in cooling.

* "A. B. C. of Iron and Steel", Purton Pub. Co.

The theory of residual strain shows that the range of stress of one kind is not as efficient in eliminating residual strain as range of stress from tension to compression. Where the range is from tension to compression extending equally beyond the natural elastic limits the reaction of the accumulation of residual strain is neutralized because of the opposite signs of its constituents, accompanied by increased internal disturbance and more rapid approach to rupture under repeated stress. Where the limiting range was tension from 20 000 to 42 000 lb. per sq. in., the systematic work on the specimen effected an accumulation of approximately 19 000 lb. of residual strain, which, added to the yield point of 24 000 lb., would give a maximum resistance of 43 000 lb. per sq. in. without plastic deformation. In other words, the effect of a tensile stress of 42 000 lb. per sq. in. on a bar under a state of residual tension of 19 000 lb. or more could not cause plastic deformation in a specimen the yield point of which is 24 000 lb., since the algebraic sum of these is only 23 000 lb. and, therefore, less than the yield point of the bar.

The fact that rupture of ductile metal under Wöhler's treatment takes place without plastic elongation has been in the past erroneously regarded as in some way indicating or establishing the fact that all ranges of stress without shock tend to deteriorate or injure sound metal. The law of conservation of energy indicates that the same amount of energy is required to produce rupture whether it is expended in small quantities at different times, or at one time in a single application. When rupture is accompanied by nearly negligible ultimate deformation, as in endurance tests under repeated stresses of intensity lower than the ultimate strength of the material, there appears to be no good reason to suppose that the total energy transformed into heat in small amounts under successive repetitions of the load in endurance tests differs from that when this deformation is as large as in the case of a single load developing the maximum strength of the specimen.

From the foregoing discussion have been drawn the following conclusions:

1.—Steel of good quality has definite limits of elasticity which can be lowered but not raised.

2.—Strain beyond these limits cannot occur without performing work on the metal tending to wear it out.

3.—Strain within these limits tends to improve the metal in elasticity, ductility, and tenacity, by elimination of the residual strains from rolling and cooling.

4.—The limit of possible improvement by repeated stress within the limits of elasticity is readily computed and accords satisfactorily with the investigations of Bauschinger and with his conclusions that, far from being fatigued by repeated stress from tension to compression within the natural elastic limits, sound metal is improved in elasticity, ductility, and tenacity.

5.—In the temporary structure the yield point is the critical limit of stress, for, if the yield point is exceeded under quiescent stress, failure is liable to result from the dynamic effect of the load in motion. The plastic deformation, moreover, may so change the configuration of a truss that failure may result. That bridge structures frequently stand up when counters are over-strained and

stretched may be accounted for by the momentary duration of the maximum stress due to the moving load.

From the theorem that volume is a kinetic invarient of elastic strain in an isotropic homogeneous solid,* the moduli, K (of lateral contraction) and F (shear), may be computed as $0.4141e$ and $0.3535E$, respectively, (before the temperature changes from the applied stress are dissipated by radiation.) Since for over-strain and small set the residual strain will be developed along the curve of greatest displacement and no over-strain takes place along the equipotential curves, F, for small over-strain, will remain numerically constant and only K and E will vary. Hence, the limit of increase in E may approximate 5% by the elimination of residual strains through repetition of stress from tension to compression within the natural limits of elasticity.

The conclusions which should guide in fixing the magnitude of working stresses the writer would summarize as follows:

1.—The working stress for a permanent structure should be kept below the natural limits of elasticity allowing properly for the effect of impact, unsymmetrical distribution of stress across the section due to faulty workmanship, and irregular form.

2.—These working stresses and proper allowances need be no lower for a range of stress from tension to compression than for stress of one kind alone.

3.—It is highly irrational to use higher working stresses for variation of stress of one kind only than for variation of stress of both kinds, because it is impossible in the actual structure for the loads to be applied after the manner of the fatigue experiments of the laboratory by varying the stresses back and forth below the normal yield point until there is temporarily an accumulation of residual strain. This procedure, contrary to any conceivable application of load in the engineering structure, constitutes the hocus-pocus of fatigue formulas; it causes needless waste in the design of bridge members subject to alternating stress, the adoption of unbalanced sections, and a deficiency of strength where the range of stress is small.

Bearing in mind that this natural elastic limit or limit of thermal proportionality hardly exceeds 22 000 lb. per sq. in for soft steel of 55 000 to 60 000 lb. ultimate strength, or 26 000 lb. per sq. in. for medium steel from 65 000 to 75 000 lb. ultimate strength, the desirability of using medium steel rather than soft steel is obvious.

In respect to columns, a fundamental defect of nearly all published column formulas lies in the disregard of twisting. An angle strut† having a ratio of $\frac{l}{r}$ equal to 140 and fitted with ball and socket bearings, is only one-half as strong as a pipe section of the same cross-sectional area and the same radius of gyration. In other words, the ordinary column formula for such a strut is liable to be in

* "Elasticity and Strength of Materials," Turner, Section 1, p. 75, see, also, Unwin, "Strength of Materials."

† A pipe strut is equally strong whether pin ended or with ball-and-socket bearing whereas an angle is 1¾ times as strong with pin as with ball-and-socket bearing when $\frac{l}{r}$ is greater than 140 although calculated the same by present theory.

error 100% in respect to the strength of these struts if the ratio, $\frac{l}{r}$, is 140 or greater. The inferior showing of steel Z-bar columns as compared with wrought-iron columns in Strobel's tests may be accounted for by the fact that Strobel made one radius of gyration much more than the other, thereby forcing the columns to fail by simple bending instead of by combined bending and twisting as in the U. S. Bureau of Standards tests.

A more careful consideration of the facts determined by test will profit the Engineering Profession, while blind acceptance of ingenious algebraic formulas founded on assumptions which disregard the vital facts of experiment may lead to serious error, lack of economy, and occasional failure.

Mr. Prichard states there are faults in the theory of flexure,* and correctly points out that shear strain is disregarded in the ordinary theory of flexure. This disregard does not increase the maximum fiber stress at the points of greatest deflection as he appears to suppose, nor does the exact theory substantiate the supposition of Marburg that the modulus, E, of Bethlehem beams is any lower for a given grade of steel than that of standard beams. Far from increasing the maximum fiber stress as computed by the ordinary theory of flexure in the restrained beam, the maximum tension is reduced both over and between the supports and a similar reduction holds true for continuous beams. The resistance to external shear may be two to four times as great per inch of section in the restrained beam as in the simply supported beam of the same cross-section. Again, as the neutral plane of the simply supported beam drops downward toward the end of the beam approximately one-tenth of the depth according to the exact theory of flexure, how can this theory be applied to a column, assuming that the neutral plane follows the axis of the piece when it does not in the beam? Moreover, since the principal tensions in the column are at right angles to the axis or cross-wise of the piece until the stage of incipient failure is approached, how can the bending of a column be compared with the bending of a beam, in which the greatest tensions are nearly parallel to the axis of the piece? Is not much of the mathematical discussion of the paper based on assumptions which are unrelated to the facts?

H. F. Moore,† Esq. (by letter).‡—This interesting paper is a welcome attempt to meet the question "What is an allowable stress for structural steel?" The author recognizes that this is very far from being a simple matter of taking some value from a set of handbook tables and applying it in a standardized formula.

One interesting point of Mr. Prichard's paper is his use of the "Useful Limit Point" (U. L. P.) rather than the yield point, or Johnson's apparent elastic limit (which involves an increase of rate of strain over the initial value amounting to 50 rather than 100 per cent. A more extended statement from Mr. Prichard as to the advantages of the U. L. P. over Johnson's elastic limit would be of value.

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* *Proceedings*, Am. Soc. C. E., November, 1925, Papers and Discussions, p. 1776.

† Research Prof. of Eng. Materials, Univ. of Illinois, Urbana, Ill.

‡ Received by the Secretary, November 27, 1925.

In connection with Mr. Prichard's suggestion as to the possibility of "remote fatigue", the writer would like to quote results not conclusive, but interesting, from some tests of I-beams made about thirteen years ago in the laboratories of the University of Illinois. Several I-beams were loaded in flexure with loads which stressed the compression flanges to the yield point. The load was continued for 107 days without causing failure of the beams, and after the first few days there was no further material increase in deformation.

Relative to "remote fatigue", the inter-relation of localized stress and corrosion are parts of a subject on which there has been very little investigation. From certain experiences with boilers, the writer believes that this is a very important subject for research by the structural engineer.

In connection with the author's discussion of fatigue tests, the writer would call attention to the results reported* by Mr. Irwin, of the Westinghouse Laboratories. Mr. Irwin developed a method of gripping the specimen in a repeated stress test under cycles of tension and compression, by which accidental bending action was reduced to a very small value. For all the metals tested thus far, Mr. Irwin finds that the fatigue limits determined by such tension-compression tests and by tests in reversed flexure are almost the same. Previous investigators from Wöhler on (including the writer) have been rather prone to assume that a tension test specimen is under pure tension. As a matter of fact it is very difficult, even under laboratory conditions, to avoid appreciable bending action, especially on a short tension test piece.

G. B. Woodruff,† M. Am. Soc. C. E.—If this paper did no more than call attention to the uncertainties of present knowledge as to the behavior of structural steel specimens and structural members under stress, and the relation of the strength of the member to that of the specimen, it should prove valuable in pointing out the need of further investigations of these matters.

In so far as safe unit stresses are concerned, the problem may be divided into three main phases:

1.—What is the "utilizable capacity" of a structural member?

2.—What requirements may be made of the test specimen that will give the most definite information as to the capacity of the structural member?

3.—What factors may be applied to the results of specimen tests to give the safe load in a structural member?

According to the author's definition, it would be permissible to stress a member to its "utilizable capacity." Evidently this is not his interpretation since he later introduces the "factor of safety," and by this factor apparently destroys his definition. Considered alone, the knowledge of the U. L. P. of a structural member would not seem to be especially valuable. Admitting that the term "proportional limit" is to a certain extent a misnomer, the deviation of the stress-strain curve from a straight line, as generally understood, is so small as to be negligible. Structural theory, especially in the case of indeterminate stresses, is predicated on the proportionality of stress and strain.

* *Proceedings*, Am. Soc. of Testing Materials, 1925.

† Asst. Engr., Modjeski & Moran, New York, N. Y.

It would seem that the structural designer is more interested in the proportional limit than in any other characteristic of the behavior of the members. If this proportional limit could be positively predetermined and the design based on the maximum possible combination of loads, no factor of safety would be required.

In the speaker's opinion, therefore, the desired result of the specimen tests is to obtain the best possible index of the proportional limit of the fabricated member. Perhaps the U. L. P. of the specimen will furnish this index, but further investigation of this point is required. That the yield point "as determined in ordinary commercial practice" gives far from satisfactory results is generally admitted. The question here is whether the trouble lies with the yield point as such or with "ordinary commercial practice." The data presented in Table 9* merely indicate errors in the operation or calibration of the testing machines. At present, there seems to be no better indication of the proportional limit of a fabricated member than the ultimate strength of the test specimen.

The speaker interprets the results of the most careful tests as indications that the proportional limit of fabricated short columns and fabricated tension members approximates 30 and 40%, respectively, of the ultimate strength of the test specimens. He believes it to be conservative when designing for maximum loads to use unit stresses of 25 and 35% of the ultimate strength of the specimens.

The proposal to vary unit stresses with the thickness of the material is a step in the right direction. Quite likely the variable in this case, however, is the amount of working the material receives in the rolling. The thickness is only a partial indication of this variable.

The author's specification for lattice† is similar to that in some recent specifications and probably results in as good design as is possible within the limits of present knowledge. Some tests, however, indicate that the reaction from the light lattice in vogue before 1907, has gone too far and that it is possible to use lattice too stiff and with too rigid connections for the maximum strength of the columns.

In conclusion, the speaker would urge that the arguments developed by the author be given full consideration, especially by those urging a revision in working stresses. The paper opens a field of interesting possibilities for the Structural Division of the Society.

R. VON FABRICE,‡ ASSOC. M. AM. SOC. C. E. (by letter).§—An examination of the various tests quoted shows that the yield point is a variable quantity if found from specimens taken from the same plate but tested in different laboratories (see Table 9). Evidently, the true elastic limit can be correctly defined only by means of the closest scientific observations.

* *Proceedings*, Am. Soc. C. E., November, 1925, Papers and Discussions, p. 1756.

† *Loc. cit.*, p. 1770.

‡ Designing Engr., Public Service Production Co., Newark, N. J.

§ Received by the Secretary, December 14, 1925.

Furthermore, Mr. Prichard's references show that the elastic limit is not a constant but a variable quantity and can be changed at will by adopting different conditions of straining. Therefore, it is evident that none of these points should be used as a criterion for the strength of any members.

The author's suggestion of basing practical formulas on the U. L. P. basis, as was adopted by the Society's Special Committee on Steel Columns and Struts, must be recognized as a step forward toward the solution of the problems dealing with the strength of structural steel.

The elevation, to higher limits, of the elastic limit shows that steel, as well as other material, will adjust itself to variable conditions and survive severe overstraining by readjustment of the grains or fibers. This behavior of steel may be compared with the characteristics otherwise attributed to the muscular development of an athlete, in whom stiff joints and muscular defects may be compared with the causes underlying auto-stresses in a piece of steel, and with whom careful working and manipulation will cause these defects to disappear as weak fibers or grains in steel recover and remain elastic after being subjected to a certain cycle of stresses. In both instances the elasticity has been raised or increased. If, however, the training or cycle of stresses is beyond the U. L. P., negative results will be obtained and permanent fatigue or failure will be the result in man and material. This behavior of structural steel under loading, adjusting itself to severe loading, accounts for the existence and apparent stability of structures which according to the recognized mathematical calculations should have failed.

Mr. Prichard shows that large variations in elastic limit exist. It would be advisable to determine by tests if a lowering of the lower elastic limit in compression is possible and if it is accompanied by a lowering of the upper elastic limit in tension; in other words, if by a cycle of stresses the elastic limit of the material under compression can be raised thereby losing a part of its resistance against tensile stresses.

The fibers of a piece of metal under tension undoubtedly have a tendency to align themselves in the direction of pull, and thus produce the increased elastic resistance (Fig. 5 (*a*) and Fig. 5 (*b*)).

Under compression the fibers tend to flow or squeeze sidewise, which causes transverse shear and consequent buckling of the specimen (Fig. 5 (*c*)). Therefore, it is fair to conclude that under tension all fibers or molecules unite in aiding each other, while under compression they tend to work against each other and squeeze and flow apart. If this conclusion is correct the elastic resistance of steel under compression cannot be increased by a cycle of stresses.

Tables 9 and 19* show great variation in the quality of the material taken from different parts of the plate or I-beam. For example, a 15-in., 38-lb. I-beam gives the U. L. P. for the web as 42 500 lb. per sq. in., while for the root it is 26 000 lb. per sq. in., and, further, the tests for the other beam sections in Table 19 show that the flange has a still different U. L. P. This condition may be due to the process of rolling whereby the web becomes more compacted than the flanges.

* *Proceedings*, Am. Soc. C. E., November, 1925, Papers and Discussions, p. 1782.

The variation in U. L. P. in the various parts of a structural shape leads to the conclusion that not only the elastic resistance, but also the moduli of elasticity of the various parts of any section may vary. Tables and references show a variation from 28 000 000 to more than 31 000 000 in moduli of elasticity for various compositions of steel; but is it not possible that the modulus of elasticity will vary in the same structural section within these limits?

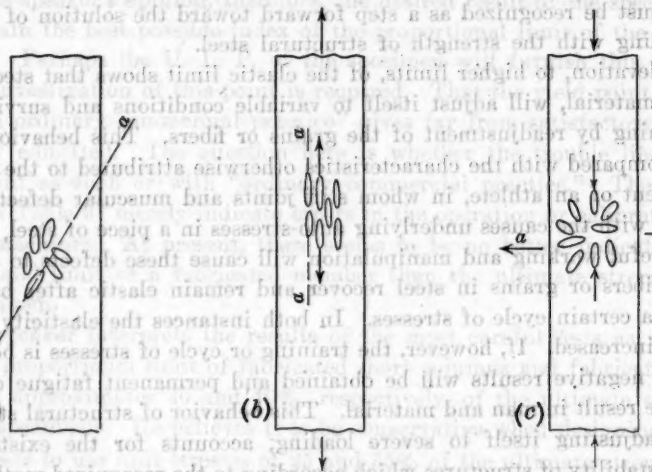

FIG. 5.—ALIGNMENT OF METAL FIBERS UNDER STRESS.

The theory of flexure is based on ideals, that is, perfectly elastic material with uniform elastic limits throughout the entire member. This condition does not exist in practice. The theory is further based on the erroneous assumption that plane sections remain plane after bending. This is not correct since the influence of the shear (which is usually ignored) prevents the originally plane cross-section from remaining plane. Further, this deformation of the section must become more pronounced if the elastic elongations of the various parts of the section vary in unforeseen and incalculable dimensions.

The rule* for finding the U. L. P. of built-up members from specimen tests may give an approximately correct value for tension members and short columns not subjected to flexure. Even tension members may be composed of materials of varying elasticity in such a manner that the resultant of the elastic properties of the different parts does not coincide with the axis of the member. This will result in incalculable auto-stresses since the internal stresses are not balanced around the axis of the section. Mr. Prichard brings this out forcefully in his appeal for an investigation to classify the various sizes of structural shapes into groups having minimum variations in their U. L. P. values. To reduce auto-stresses to the minimum and to manufacture steel uniform in quality and in elastic properties is an ideal which can be hoped for but which probably never will be attained.

To classify the various groups of steel according to the variations of the U. L. P., it is necessary first to establish by a series of tests the variation of

* *Proceedings*, Am. Soc. C. E., November, 1925, Papers and Discussions, p. 1758.

auto-stresses, U. L. P.'s and other elastic properties in various rolled shapes and in their different parts, and also to give due consideration to the different rolling processes used by the various steel mills.

From Mr. Prichard's investigations it does not seem possible that built-up members will ever consist of materials of entirely the same class or quality. Fortunately, however, in built-up members the materials with greater elastic resistance seem automatically to go into the extreme fiber; for instance, a Bethlehem H-column or a girder reinforced by two plates will have the more reliable material in the plates. The same holds true for plate girders with cover-plates; Table 9 shows that the edge portions of plates are stronger than the interior ones.

Beams.—To consider the influence of shear, Mr. Prichard recommends* adding a percentage, P_e, to the intensity of stress as computed from the ordinary flexure theory and advocates the use of Equations (20), (21), and (22).† He then gives‡ the possible error of these equations as up to 50 per cent. It is difficult to understand why the old and approved theory of flexure should be revised for an empirical formula afflicted with 50% factor of error.

As even good steel is not a perfectly elastic medium; as there is the possibility of variable elastic properties of different parts of one and the same shape; and as the designers of steel structures are not assured of the elastic properties of the various parts of built-up members, it is logical to assume that all these unknown factors probably bring about a greater pronounced warping of plane cross-sections than the hitherto neglected influence of the shear.

Shear.—Of course, for short beams with heavy concentrated loads provision should be made to bring this load safely over the supports without distortion of the local parts. In such cases the flanges should be supported by milled stiffeners. Web sections and reinforcement in excess of ordinary theory should be provided as recommended by Mr. Prichard.§

Columns.—Under Paragraph (*m*) of the Synopsis‖ it is stated that the U. L. P. of a light H-column was 35 000 lb. per sq. in., while that of an extra heavy H-column of the same material was only 19 000 lb. per sq. in. According to Table 11,¶ twelve tests were made with these extra heavy columns and only three with light ones. The writer does not believe that these tests should be used for final deductions.

All the tension tests of Table 11 show an increase of unit strength for the heavier columns with the exception of tests for No. 5 which show the reverse. It is recommended that light and heavy H-columns be re-tested in order to determine whether the cross-section of the heavier columns was properly proportioned and the stress evenly distributed over the entire area, rather than to arrive at the conclusion that the U. L. P. of a heavy column is only

* *Proceedings*, Am. Soc. C. E., November, 1925, Papers and Discussions, pp. 1776–1781.
† *Loc. cit.*, pp. 1777, 1779.
‡ *Loc. cit.*, p. 1781.
§ *Loc. cit.*, p. 1782.
‖ *Loc. cit.*, p. 1737.
¶ *Loc. cit.*, p. 1760.

about 60% that of a light column of practically the same shape as shown in Paragraph (*m*).

Heavier sections of the same or nearly the same over-all dimension have a smaller radius of gyration, which confirms the conclusion that an increase of area without an increase of outer dimensions is not an economical solution. Therefore, Mr. Prichard is correct in his statement that heavier columns possess a lower U. L. P. than lighter ones; but the variation from 19 000 to 35 000 lb. per sq. in. seems to be too large in view of the tests recorded in Table 11.

By his research work Mr. Prichard has added most valuable material as to the theory of columns, but his suggestion to classify columns into three divisions—short, medium, and long—does not simplify the already very complex problem of column design. What is needed for designers is simplification, not theoretical refinements and consequent complications as to the selection and classification of columns.

Equations (1),* (6)†, and (11)‡ have been derived assuming frictionless hinges at the ends, a condition which does not exist in practice. For columns restrained at both ends, the recommendation is made to use as l of the equation a value from 0.67 to 0.75 of the actual length. The wide range of this recommendation eliminates all refinements gained in developing Equations (6) and (11).

The author's classification of columns cannot be termed a happy solution since as the first two, short and medium, are made dependent on the U. L. P., while the long columns are independent of their U. L. P.'s or of any other critical stress mentioned.

Table 12§ cannot be readily understood, nor do the illustrations of its use‖ shed any light on this subject. It would be helpful if the author would add further discussion of this part to clarify his data and increase the usefulness of the paper.

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* *Proceedings*, Am. Soc. C. E., November, 1925, Papers and Discussions, p. 1761.
† *Loc. cit.*, p. 1764.
‡ *Loc. cit.*, p. 1765.
§ *Loc. cit.*, p. 1762.
‖ *Loc. cit.*, p. 1767.

STREAM POLLUTION

A SYMPOSIUM

Discussion*

By H. A. Whittaker, Esq.

H. A. Whittaker,† Esq. (by letter).‡—Dr. Frost presents in his paper§ a clear picture of the past and present scientific investigations of the United States Public Health Service on stream pollution and certain allied problems. The program of investigations now in progress, which include intensive studies on the natural purification of streams, the subject of sewage treatment, and water purification, covers the fundamentals of the problem, and embraces most of the agencies involved in the protection of man against water-borne diseases. The fact that these investigations are being conducted under both experimental and natural conditions increases the possibility of obtaining accurate information on the basic laws relating to these subjects and their application to practical conditions. Since the results of these investigations have a nation-wide application, this field of study is logically one for the Federal Government to sponsor although certain problems may be undertaken in co-operation with the various States that are equipped to carry on similar activities.

It is now recognized that where public water supplies are involved, there is only one attitude toward this question and that is that the pollution of streams and lakes used as water supplies must be so limited as not to impair the safety or palatability of any water supply obtained from the same stream or river. The other condition, that the stream or lake is not used as a source of water supply, is a phase of the problem which is more difficult of solution. Here, on the one hand, is the value of streams and lakes for recreational purposes—boating, bathing, fishing, etc.—admittedly of great value to the public but very difficult to place on a monetary basis; and the use of streams and lakes for commercial fishing, the value of which can be quite definitely established. On the other hand, is the relatively great expense of constructing and operating sewage treatment works to which municipalities and industries would be subjected if all sewage and waste previously discharged into inland waters were to be treated.

It must be admitted quite candidly that except where water supplies and bathing beaches are affected, or cattle are exposed to sewage-polluted streams

* Continued from January, 1926, *Proceedings*.
† Director, Div. of Sanitation, Minnesota State Board of Health, Minneapolis, Minn.
‡ Received by the Secretary, December 21, 1925.
§ *Proceedings*, Am. Soc. C. E., November, 1925, Papers and Discussions, p. 1810.

and lakes, the health aspect of this problem is relatively slight. In most cases the cost of constructing and operating sewage treatment works greatly exceeds the value of commercial fishing in inland waters. In many instances bathing beaches can be established in waters not subject to sewage pollution, or such facilities can be provided by the construction of artificial bathing places. Cattle may be affected by contact with sewage polluted waters, but the fact remains that these same cattle are probably in greater danger of contracting disease from infected animals that are pastured up stream on the same watershed than from municipal sewage. It would seem that, in most cases, cattle can be more surely protected against all forms of water-borne diseases by fencing off the streams and providing the animals with clean drinking water from other sources than by the installation of sewage treatment works.

Apparently, where public or industrial water supplies are not affected, it must either be frankly admitted that the recreational value of streams and lakes is not commensurate with the cost of sewage treatment and the lakes and streams not used as sources of water supply, must be considered simply as places for the natural disposal of sewage, or the intrinsic worth of preserving these lakes and streams in their natural condition must be set so high that there can be no question as to the soundness of a policy of treating practically all sewages and industrial wastes to such a degree that the condition of the stream or lake will be practically what might be expected if no sewage or wastes were discharged into it. Just what will be the final outcome only future developments will reveal. Already powerful organizations, such as the Isaac Walton League, are endeavoring to mould public opinion and create a public demand for protective measures against stream pollution.

The Minnesota Department of Health is taking the attitude that no drastic legislation should be enacted until a comprehensive sanitary engineering survey has been undertaken on the streams and lakes throughout the State, and until rather detail information relative to actual conditions has been obtained. After this has been done, it should be possible to outline a policy which would have a remedial effect so far as the State of Minnesota is concerned.

Minnesota contains the head-waters of several important rivers which flow into or adjacent to other States or provinces. Two adjoining States have already entered formal complaint regarding the pollution of interstate rivers. The Wisconsin State Legislature, during its 1925 session, appointed a committee to take up with the Minnesota Legislature, then in session, the question of the pollution of the Mississippi River. As a result of this action the Minnesota Legislature also appointed a committee to investigate the pollution of the boundary waters between Minnesota and Wisconsin. The two Committees have held a joint meeting and have decided to co-operate in their study of the problem which is of common interest to both States. The Committees have jointly requested the U. S. Public Health Service to detail a sanitary engineer to co-operate with the sanitary engineers of the Departments of Health of Wisconsin and Minnesota in working out a plan of scientific study of the Mississippi River in relation to stream pollution. The Cities of Minneapolis and St. Paul have also offered to co-operate in this study which is

of interest to them in determining a future course to follow in regard to sewage disposal. The interests of the parties involved are so varied that any co-operative work, to be acceptable to all, must necessarily be directed by some disinterested and thoroughly competent agency such as the U. S. Public Health Service.

It is hoped that the splendid work already being conducted by Dr. Frost and his associates in the Service will be continued and expanded so as to insure the just and equitable solution of the stream-pollution problem in the United States in all its phases.

THE EFFECT OF MOISTURE ON CONCRETE

Discussion*

By W. K. Hatt, M. Am. Soc. C. E.†

W. K. Hatt,‡ M. Am. Soc. C. E. (by letter).§—Experiences related by Professor Smith‖ show the effects of expansion of the lower part of a concrete pipe under extreme conditions. It will be necessary to know something more about the manufacture of pipe in order to judge whether or not these examples are representative. In a paper published by the U. S. Bureau of Public Roads, entitled "Concrete Pipe for Use in Irrigation",¶ warning is given against laying such pipe in sandy soil that is hot and dry. It would appear that extra precaution should be taken in the curing of pipe in climates such as that of Arizona, because it is well known that surface crazes result from improper curing with extreme exposure. These conditions indicate the desirability of using mesh reinforcement.

During May, 1925, tests were made at Purdue University on two 30-in. concrete pipes, machine made, of 1 to 5 mix. The pipes were 3½ in. thick and one was reinforced with mesh. They had been steam-cured and air-dried for approximately three months. After submerging them in water to one-third of their depth for 5 days, measurements were made around the inner and outer surfaces with a Berry strain gauge. No cracks were apparent to the unaided eye. After this test the pipes were loaded in the testing machine and crushed; each pipe tested 2 500 lb. per lin. ft.

There is usually a difference in the behavior of new and old concrete under test. In old concrete small fissures appear but the pipe continues to take more load, whereas in newer concrete the pipe seems tougher and the first visible crack corresponds more nearly to the maximum load.**

The behavior of concrete under moisture, temperature, and load has been discussed at greater length in a paper entitled "Researches in Concrete."**

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* Discussion on the paper by W. K. Hatt, M. Am. Soc. C. E., continued from January, 1926, *Proceedings*.

† Author's closure.

‡ Prof. of Civ. Eng., School of Civ. Eng., Purdue Univ., Lafayette, Ind.

§ Received by the Secretary, December 28, 1925.

‖ *Proceedings*, Am. Soc. C. E., January, 1926, Papers and Discussions, p. 164.

¶ *Bulletin No. 852*, U. S. Bureau of Public Roads.

** *Bulletin No. 11*, Eng. Experiment Station, Purdue Univ., November, 1925.

ELIMINATION OF ODORS FROM GARBAGE DISPOSAL WORKS

Discussion*

By Samuel A. Greeley, M. Am. Soc. C. E.†

Samuel A. Greeley,‡ M. Am. Soc. C. E. (by letter).§—Mr. Lewis‖ properly emphasizes the need for beginning studies at the garbage pail itself and including both collection and disposal for the control and proper elimination of odors.

Mr. Osborn¶ comments on the relation between the location of a garbage disposal plant and the cost of haul. This is undoubtedly an important item. As indicated in the paper,** zoning boards have recognized certain odor-producing industries by outlining so-called nuisance and semi-nuisance districts. Presumably the amount of odor control for such industrial establishments is related somewhat to their location. Just how far such practical considerations should influence the design and cost of a garbage disposal plant must be determined for each special case. The writer recently investigated such a situation in a city of 140 000 population. An old established site, well isolated, was available for the garbage disposal plant. The use of another site would have permitted the collection work with one less wagon, effecting a reduced annual cost amounting to about $5 000. However, it was decided that the security of the existing site against nuisance hazards was worth the extra cost.

Mr. Osborn's comments on the fundamental features of design and his statement that engineers are not using available experience, are sound. However, records of operating experience with garbage disposal plants are not as well advanced as other branches of sanitary engineering work. Undoubtedly, a great deal of such experience has been had, but relatively little of it has been published in a definite way, as yet, with reliable yardsticks. In comparison with the large sums that have been expended by municipalities in testing sewage disposal processes, there is much yet to be done in the field of garbage disposal. Mr. Osborn's statement, therefore, that the disposal of garbage in a sanitary manner is a function requiring study on which municipal officials and engineers should look with the same interest as they do on other municipal problems, is a good summary of the situation.

* Discussion on the paper by Samuel A. Greeley, M. Am. Soc. C. E., continued from August, 1925, *Proceedings*.

† Author's closure.

‡ (Pearse, Greeley & Hansen), Chicago, Ill.

§ Received by the Secretary, January 4, 1926.

‖ *Proceedings*, Am. Soc. C. E., August, 1925, Papers and Discussions, p. 1177.

¶ *Loc. cit.*, p. 1178.

** *Loc. cit.*, May, 1925, Papers and Discussions, p. 832.

MEMOIRS OF DECEASED MEMBERS

NOTE.—Memoirs will be reproduced in the volumes of *Transactions*. Any information which will amplify the records as here printed, or correct any errors, should be forwarded to the Secretary prior to the final publication.

FRANK VALENTINE ERHARD BARDOL, M. Am. Soc. C. E.*

DIED APRIL 9, 1925.

Frank Valentine Erhard Bardol was born on June 12, 1869, at Erie, Pa. He received his early education in the public schools of Buffalo, N. Y., where his parents moved when he was a child. At the age of 16 he entered Cornell University as a student in Civil Engineering and was graduated with the degree of Civil Engineer in 1889.

In July, 1889, Mr. Bardol began his engineering career as Draftsman and Principal Assistant Engineer with the Lehigh Valley Railroad Company, at Buffalo, N. Y., under Division Engineer, S. T. M. B. Kielland, M. Am. Soc. C. E., where he was engaged in terminal improvements, including the construction of docks, warehouses, and yards at Buffalo, as well as at Duluth, Minn., and Chicago, Ill. He remained in the service of the Lehigh Valley Railroad Company until the end of 1891.

In January, 1892, he was appointed to and accepted the position of Assistant Engineer in the Bureau of Engineering, Department of Public Works at Buffalo, in direct charge of the construction and maintenance of bridges, street pavements, and also of general surveys and other miscellaneous work in the Bureau. He held this position until the end of 1897.

In January, 1898, at the age of 29, Mr. Bardol was appointed City Engineer of the City of Buffalo, and in this capacity was Chief Engineer of the Bureau of Engineering, Department of Public Works, in charge of the supervision of all construction and maintenance of public work in the city, except water. During his term of office as City Engineer, he was confronted with a number of the city's municipal problems, among which were the increase of the navigability of the Buffalo River; the solution of the railroad terminal problem for the city's various incoming lines; the Hamburg Canal nuisance; and various harbor improvements, which he was instrumental in obtaining through his efforts before the River and Harbors Committee at Washington, D. C.

On the expiration of his term of office as City Engineer of Buffalo at the end of 1901, Mr. Bardol retired from public service and immediately opened an office in Buffalo as Consulting Engineer and afterward as Construction Engineer. The years that followed in his comparatively short life were eventful in the extreme; his natural gifts along many different lines were outstanding not only in the formation of successful construction companies, but also as a keen investor which enabled him in a few years to become a multimillionaire.

* Memoir prepared by F. J. Tresise, M. Am. Soc. C. E.

In 1902, Mr. Bardol formed the Eastern Construction Company, and as its President built a number of asphalt streets in Buffalo and adjacent towns. This Company was dissolved at the end of two years. In 1904, he organized the Eastern Concrete Steel Company, and as President built many structures, among which were a number of bridges, viaducts, and subways along the line of the New York Central and Hudson River Railroad; also similar structures for the Lehigh Valley and Delaware, Lackawanna and Western Railroads, the State Normal School in Buffalo; the Buffalo City Hospital; and many other large buildings for Cornell University; Syracuse University; the Eastman Kodak Company, of Rochester, N. Y., and the Larkin Company, of Buffalo.

In 1911, he became associated with the Rock Asphalt and Cement Corporation of Buffalo, and as Vice-President and General Manager was in charge of the construction of many millions of dollars of public improvements throughout the city. He was engaged in this capacity until his death.

In 1916, Mr. Bardol added to his many other interests the Erie Beach Amusement Park. In connection with the ownership of the Park, he purchased the Niagara Ferry and Transportation Company, operating between Buffalo and Fort Erie, Ont., Canada, across the Niagara River; the Buffalo and Fort Erie Railway Company, operating between Fort Erie and the Park; and a steamship line operating summer excursion boats from the foot of Main Street, Buffalo, to the Lake Dock at Erie Beach Park. This large enterprise, which for years had continued as a failure was, through his executive ability and business acumen, put on a highly paying basis within the nine years of his ownership and personal management, and enlarged and improved to such an extent that at the time of his death it ranked second to none in the United States.

In describing the career of Mr. Bardol the editor of a local paper stated:

"Frank V. E. Bardol was a most successful, if not conspicuous figure in the life of Buffalo. As Vice-President and General Manager of one of the two large local Paving Companies, he had the spending of millions of the taxpayers' money in the course of 25 years, but the politics inevitably connected with such a business he apparently left to his associates.

"Mr. Bardol was a thoroughly practical engineer with a keen business mind, and was one of those men who could make money out of his hobbies for he made a hobby of operating the Erie Beach, which he bought out of the Bankruptcy Court."

He was married to Katharine M. Bott, of Buffalo, on May 11, 1910, who, with three children, Franklin V. E., Robert George, and Edgar John, survives him.

Mr. Bardol was of a kindly disposition, and was eager to help those who were worthy of assistance. This was fully shown in his $250 000 endowment for scholarships in the University of Buffalo, and many other smaller gifts.

He was a member of the Buffalo Club; Buffalo Athletic Club; the Rotary Club; Buffalo Orpheus; Washington Lodge, F. and A. M.; the Buffalo Consistory; Ismailia Temple; and the Buffalo Fraternal Order of Eagles.

Mr. Bardol was elected a Member of the American Society of Civil Engineers on May 2, 1900.

HENRY CLARENCE HODGKINS, M. Am. Soc. C. E.*

DIED NOVEMBER 8, 1925.

Henry Clarence Hodgkins was born at Carthage, N. Y., on December 12, 1854, the son of Ezra and Amelia (Fuller) Hodgkins. His mother's ancestry has been traced to Edward Fuller, who came to America in the *Mayflower*. The Hodgkins and Fuller families belong to a group of miners and manufacturers who were attracted to Northern New York by the mineral wealth and splendid water-power facilities of the Black, Grass, and Oswegatchie Rivers. The Towns of Fullerville and Rossie were the sites of foundries and furnaces operated by Fuller Brothers, one of whom, Ashbel, was the maternal grandfather of Henry Clarence Hodgkins.

Mr. Hodgkins was graduated from Union College with the degree of Civil Engineer in 1875. He then began a long and useful career as Engineer in the survey and improvement of the islands of the St. Lawrence River, including the following: Thousand Island Park, Westminster, Round Island, Hillcrest, Grenell Island, and Central Parks. He also did considerable landscape work in parks and cemeteries at Watertown, N. Y., and extensive surveys in a long litigated dispute between the Rossie and Karney Ore Mines.

In 1880, he became Assistant Engineer on the Water Power Pumping Station at Watertown, and in 1881 held the position of Topographer on the New York and Boston Inland Railroad surveys. In 1882, he designed and supervised the construction of three masonry arch bridges across the Grass River in St. Lawrence County, New York, with an aggregate length of 850 ft., and also served as Transitman on the location of the Carthage and Adirondack Railroad.

From 1883 to 1885, Mr. Hodgkins served as City Engineer of Watertown, and in this capacity designed and supervised the construction of the Court Street Bridge and approaches with a clear span of 170 ft. This bridge in 1921 was replaced by a double-deck concrete structure.

In 1884, he entered into partnership with Hinds, Moffett, and Company, which a short time afterward became the firm of Moffett, Hodgkins, and Clarke. As the engineer member of this firm, he had charge of the plans and construction of seventy-five water-works including those at Duluth, Minn., Butte, Mont., and Peoria and Galesburg, Ill. In connection with the firm, he had supervision of the surveys of the Norwood and Massena Railroad, the Kinderhook and Hudson Railroad, and took personal charge of the location and construction of the Gouverneur and Oswegatchie Railroad. The firm also acted as Engineers of the Lake Street Elevated Railroad in Chicago, Ill., installed and operated three gas plants and ten central station electric lighting plants, and installed one of the first central alternating-current plants at Rochester, Minn., in 1887.

Mr. Hodgkins served as President of the Montreal Water and Power Company from 1890 to 1893 and until 1900 was President of the Electric Engi-

* Memoir prepared by William M. Booth, Esq., Syracuse, N. Y.

neering and Supply Company, manufacturers on a large scale of electric lighting supplies. At this time he was also interested in the appraisal, arbitration, negotiation, and sale of the properties of many water companies to towns.

In 1902, he served as Engineer for the Town of Waterloo, Ont., Canada, in the construction of its water-works, and, also, as Engineer for the Village of Carthage, N. Y., in obtaining its water supply by gravity from the Adirondacks, fifteen miles away.

From 1904 to 1907 Mr. Hodgkins was employed in connection with the transmission lines of the Northern Ontario Light and Power Company, Limited, and as Consulting Engineer for the City of Berlin, Ont., Canada, in the installation of electric pumping, filtration, and a concrete water tower. He also served as Consulting Engineer for the City of Brantford, Ont., on the revision of its distribution system and increased water supply, including reservoir and electric pumping; as well as Engineer for the Galion Water Company, for an increased water supply and a pumping plant. From 1907 to 1914, Mr. Hodgkins was engaged in general consulting work.

From 1923 until his death he held the position of Engineer for the City of Cortland, N. Y., studying the remodeling and enlarging of the water distribution system, the new water supply, the question of pumping by electric *versus* steam power, and the designing and constructing of a complete new pumping station. In 1914, he served as Engineer for the proposed suburban water supply at Toronto, Ont., Canada, and as Consulting Engineer for the City of Kitchener, Ont., for its additional water supply, pumping mains, and reservoir. Mr. Hodgkins also served as Engineer for the Village of Manlius, N. Y., from 1918 to 1921, on its additional water supply works, electric pumping station, and conduit.

On June 11, 1889, Mr. Hodgkins was married to Mary Ida Follett, of Newark, Ohio, and in the autumn of that year they removed to Syracuse, N. Y., where they have since resided. Two of their three children are living, Henry Follett Hodgkins and Marian Hodgkins Baxter.

In June, 1925, Mr. Hodgkins joined the surviving members of his Class at Union College in the celebration of its Fiftieth Anniversary. This was a very happy occasion. He was a member of the Technology Club of Syracuse and was elected a member of the American Water Works Association on April 18, 1888.

His contributions to the literature of his profession were made through papers presented at the annual meetings of the Societies of which he was a member. He was an authority on franchises, and this subject was the title of a much discussed paper before the American Water Works Association, in Cincinnati, Ohio, in 1915. The Syracuse Technology Club received his hearty support and co-operation. He attended the meetings with great regularity, contributed papers, and served on its committees. His knowledge of the topography of Central and Northern New York always surprised his associates.

For nearly a quarter of a century Mr. Hodgkins was a member and officer of the First Presbyterian Church of Syracuse, and to these duties he brought the dignity, integrity, and accuracy of the engineer. As a companion he was

genial, fond of anecdote, and ready to draw from a broad personal experience and careful reading. His interest in the comprehensive physical development of the City of Syracuse was unusual. His civic pride was of the highest order. In his death, the Engineering Profession has lost a useful member. He will be missed by many friends.

Mr. Hodgkins was elected a Member of the American Society of Civil Engineers on June 6, 1911.

JOHN HENRY MADDEN, M. Am. Soc. C. E.*

DIED JANUARY 22, 1925.

Within twenty-seven years after he had received his degree as Civil Engineer at Columbia University, New York, N. Y., John Henry Madden became Chief Engineer of the Transit Commission, after an active career in the supervision of construction of the New York underground railroads which made him an expert and well qualified him for such a responsible position. Few engineers can equal such a record of achievement in a similar period of time. His untimely death cut short a career which promised a brilliant future.

Born in New York, N. Y., on October 18, 1875, John Henry Madden received his preliminary education in the public schools and then entered Columbia University, from which he was graduated in 1897 with the degree of Civil Engineer. His first work after his graduation was with William Barclay Parsons, Hon. M. Am. Soc. C. E., who designed and directed the construction of New York's first subway. Young Madden worked as a Surveyor and Draftsman for the new underground railroad and became so interested in the work that he made it virtually his life's pursuit.

For only a few years did he interrupt his work on underground railroad construction. He devoted two years to service with the United States Geological Survey as Field Assistant on topography and another year to engineering design and construction under Charles W. Leavitt, M. Am. Soc. C. E., to whom he was Assistant Engineer. In 1901, he returned to his first love and became Assistant Engineer on the construction of a section of the first subway in Broadway between 60th Street and 82d Street, under the Board of Rapid Transit Railroad Commissioners of New York. When he had been on this work for six years, a change in the law replaced the Board of Rapid Transit Railroad Commissioners with the Public Service Commission for the First District. In 1907 he was transferred to the staff of the new Commission with the title of Assistant Engineer and remained in its employ and on the same kind of work for eleven years.

In those eleven years Mr. Madden showed the stuff of which he was made. He handled the most difficult and dangerous kind of subway construction, always with credit to himself and to the great satisfaction of his superior officers, among whom were Alfred Craven and Daniel L. Turner, Members, Am. Soc. C. E., and Robert Ridgway, Past-President, Am. Soc. C. E., each

* Memoir prepared by Le Roy T. Harkness, Esq., and Frederick L. Cranford, M. Am. Soc. C. E.

in turn the Chief Engineer of the Commissions conducting subway construction. Included in this period of Mr. Madden's activity were supervision of extensions of stations in the first subway between 157th Street and Dyckman Street, Manhattan—a delicate piece of work because it had to be done during the daily operation of the subway trains; the construction of four sections of the new subways—Dual System—in Lexington Avenue between 53d and 104th Streets, Manhattan; the construction of two more sections in Park Place, Beekman and William Streets, Manhattan; and four more sections in Flatbush Avenue, Brooklyn, between Atlantic Avenue and Malbone Street. All this was difficult underground construction which taxed the skill of the most experienced engineers. His work on the William Street Subway is historic in its success in underpinning lofty buildings and avoiding damage to abutting property.

In 1917, as a reward for his faithful and efficient services, Mr. Madden was appointed by the Public Service Commission to be Engineer of Rapid Transit Operations, in charge of expenditures and distribution of revenues under the operating contracts of the subways of the Dual System. The World War brought the United States into armed conflict in 1917, and the Government called many engineers into its service. Mr. Madden was one of them, receiving and accepting in the summer of 1918 an appointment from the War Department as Assistant Director of Construction of Nitrate Plants. The Director was Frederick L. Cranford, M. Am. Soc. C. E., and in association with him, Mr. Madden was in charge of the construction of the three Government nitrate plants at Muscle Shoals, Ala., and Toledo and Cincinnati, Ohio.

At the conclusion of the war Mr. Madden became a member of Mr. Cranford's Company—Frederick L. Cranford, Inc.—where he remained until 1921. In that year the Transit Commission was created by law, and empowered to make a valuation of all the street railroad properties in New York City. The Commission chose Mr. Madden for this work and appointed him its Valuation Engineer. He organized the work and made a complete valuation of the subways, elevated roads, and street surface lines, submitting to the Commission a monumental report, which is one of the greatest of its kind and is recognized as a model throughout the United States. In 1924, the Transit Commission appointed him its Chief Engineer, a position which he held at the time of his death.

Mr. Madden was highly talented and in addition was gifted with a personality which attracted warm friendships. Great ability in an engineer combines many attributes—technical knowledge, practical experience, common sense, courage, vision, fairness, scientific honesty, executive ability—all of which John Madden had, together with a positive and forceful directness which, although outspoken, never offended. Where he was known he was always respected and loved, and the sense of loss when his death occurred was marked by unusual and spontaneous expressions of regret containing warm tributes to his character and ability.

Mr. Madden was elected a Junior of the American Society of Civil Engineers on October 7, 1902, an Associate Member on May 4, 1904, and a Member on June 16, 1919.

LINDLEY MARSHALL RICE, M. Am. Soc. C. E.*

DIED AUGUST 21, 1925.

Lindley Marshall Rice, the son of James Walker and Bell (Wilson) Rice, was born at Lawrenceburg, Ky., on September 1, 1868. His forebears were of hardy pioneering Scotch-Irish stock and were among the first to cross the Appalachian Mountains to settle in Kentucky. His parents were slave holders at the time of the Civil War.

Little is known of Mr. Rice's boyhood, but his technical education was obtained at the Georgetown College, Georgetown, Ky., where he took a scientific course.

On leaving college in 1888, he went to the Northwest and began the practice of engineering in the State of Washington. His first work consisted of assisting in the construction of a slaughter-house for the firm of Rice and Gardner. When this work was completed, Mr. Rice found employment in the office of Mr. A. Gardner, then City Engineer of Seattle, Wash., as Chainman and Rodman, and he was also engaged in general municipal engineering work. In the spring of 1889, he served with the Fairhaven and Southern Railway Company of Washington as Rodman and was afterward promoted to the positions of Leveler and Transitman. He remained with this Company until the fall of 1890.

At this time the Fairhaven and Southern Railway Company was merged with the Great Northern Railway Company, and Mr. Rice was engaged as Assistant Engineer in charge of construction on the Seattle-Montana Branch, under the late William Parsons Watson, M. Am. Soc. C. E. Although he was only 22 years of age, he carried out the work efficiently.

On the completion of this project, Mr. Rice was transferred to the main line of the railroad, in the Cascade Mountains, where he served as Assistant Engineer, reporting to John F. Stevens, Hon. M. Am. Soc. C. E. This work was finished in the early part of 1893, and he then engaged in private practice for two years.

In 1895, Mr. Rice was employed on the Kalso and Slocan Railroad in British Columbia as Assistant Engineer, but early in 1898 entered the service of the Canadian Pacific Railway Company as Locating Engineer and Division Engineer on Construction. He remained on this work until June, 1900, when he resigned to enter private practice in Seattle. He devoted the following year to the location and construction of logging railroads and subsequently, until 1907, built nearly all the railroads of this type that were constructed in the State of Washington.

In 1905, he organized the Klickitat Irrigation and Power Company and served the Company at various times as Chief Engineer, General Manager, and Director. In 1907, he organized the L. M. Rice and Company, an engineering and contracting corporation, and was at various times its President, Chief Engineer, and General Manager. This Company carried out various

* Memoir prepared from information furnished by L. M. Rice, Jr., and from data on file at the Headquarters of the Society.

contracts, including the completion of about 100 miles of the Kettle Valley Railroad between Midway and Penticton, British Columbia. It had as its principal object the irrigation of arid lands, and as Mr. Rice had studied this subject thoroughly he was recognized as an authority in matters relating to irrigation.

He was connected with many of the large engineering projects of the West and was the first to conceive the idea of irrigating the Horse Heaven Plateau, a vast stretch of level, fertile land located in Southeastern Washington which on account of scarcity of water in summer had not been settled. The Horse Heaven Irrigation District was organized to take over the water rights on the Klickitat and Big Muddy Rivers for use in irrigating the Plateau and Mr. Rice became Chief Engineer of the District.

From 1915 to 1918, his great interest in reclamation work led to his appointment as Chairman of the Executive Committee of the Western States Reclamation Conference, and he was also active in the work of the Washington Irrigation Institute.

During his tenure of office with the Western States Reclamation Conference, Mr. Rice was in constant contact with many of the large Western projects. At this time he also took an active interest in the Wenatchee Heights Irrigation District and the Yelm Irrigation District, both of which were in the State of Washington. In 1918, he served as Chief Engineer of the Quincy Irrigation District.

In recent years, he had spent a great part of the winter months in Eastern cities in an untiring effort to secure financial support for the Horse Heaven Irrigation Project. This work weakened his vitality, and it was shortly after his return to Seattle from one of these trips that his sudden death from apoplexy occurred.

Mr. Rice was married at Edmonds, Wash., on September 9, 1891, to Adelaide E. Leonhardt who, with two daughters, Ophelia and Miriam E., and three sons, Lindley M., Jr., Carleton L., and James W., survives him.

Mr. Rice was elected a Member of the American Society of Civil Engineers on May 31, 1916. He was also a member of the Rainier Club of Seattle and the American Geographical Society.

RICHARD EDWARD PHILLIPS, Assoc. M. Am. Soc. C. E.*

DIED AUGUST 8, 1925.

Richard Edward Phillips was born in Lockport, N. Y., on November 23, 1880, and while still in his early youth moved to Buffalo, N. Y. At the age of 17 he terminated his attendance at High School and started his engineering career in the employ of Mr. Henry L. Lyon, a civil engineer, engaged in lot surveying.

In 1899, Mr. Phillips entered the employ of the New York Central Railroad Company at Depew, N. Y. It was in the shops of this Company that he acquired a knowledge of practical mechanics.

* Memoir prepared by F. P. Williams, M. Am. Soc. C. E.

In 1902, he accepted an appointment as Chainman in the Engineering Department of the State of New York. At that time the State was extensively engaged in work on the Barge Canal and highways preliminary to the large scale construction which later was undertaken. Mr. Phillips was engaged in highway work for about a year when he was transferred to work on the Barge Canal, with which he was identified for the remainder of his life.

He rose in the Department of the State Engineer and Surveyor from Chainman to Supervising Engineer, and for some years prior to his death held the title of Engineer of Claims. Until the Barge Canal was completed, he was actively engaged in the design and supervision of important features of that project. Later, many large damage claims resulting from extra compensation demanded by contractors and by private property and water-power owners were filed against the State. Mr. Phillips' duties consisted of the investigation and technical defense of these claims.

It was in the latter connection that he displayed his unusual ability. His exceptional knowledge of the relation of law to engineering and his information relative to the cost of construction and water power values made his services highly valuable to both the Engineering and Legal Departments of the State of New York.

Mr. Phillips' health failed about a year prior to his death and he was granted a leave of absence from his duties in the hope of effecting his recovery. All efforts toward this end were unavailing, and he died at Saranac Lake, New York, on August 8, 1925.

He was married in 1904 to Emma Christine Reed, of Albany, N. Y., by whom he is survived, with two daughters, Mrs. Louise Carey and Miss Marion Phillips.

Mr. Phillips was elected an Associate Member of the American Society of Civil Engineers on May 7, 1913.